1. We have six ropes. Each rope has its ends labeled A and B. We randomly select the A ends, two at a time, and tie them together. Perform this similarly for the B ends. What is the probability that we have a single ring after performing this?

   Solution. More generally, suppose we play this game with $2n$ ropes. Denote B_{2n} as the event of having a single ring after all operations. Denote L as the event that rope 1 forms a loop with any other rope. Then we have

$$\begin{aligned} P(B_{2n}) &= P(B_{2n}|L)P(L) + P(B_{2n}|L^c)P(L^c) \\ &= 0 \cdot \frac{1}{2n-1} + P(B_{2n-2})\frac{2n-2}{2n-1} \end{aligned}$$

   If we let $p_{2n} \stackrel{\text{def}}{=} P(B_{2n})$, we now have a recursive relation $p_{2n} = \frac{2n-2}{2n-1}p_{2(n-1)}$, with initial condition $p_2 = 1$. Iterating this gives us

$$p_{2n} = \prod_{k=2}^{n} \frac{2k-2}{2k-1}$$

   Hence

$$p_6 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{15}$$

   □

2. Container A initially contains 1000 green and 3000 red apples and container B contains 3000 green and 1000 red apples. We randomly choose 2000 apples from container A and place them in container B. One apple is then chosen at random from container B. What is the probability that it is green?

   Solution. Let F be the event we choose green in container B, and let E_k denote the event that we bring over k green and $3000 - k$ red from container A. Then

$$P(F) = \sum_{k=0}^{1000} P(F|E_k)P(E_k) = \sum_{k=0}^{1000} \frac{3000+k}{6000} \cdot \frac{\binom{1000}{k}\binom{3000}{2000-k}}{\binom{4000}{2000}} = \frac{7}{12}$$

   □

3. Archers A and B are successively shooting at a target, with A going first. Suppose the probability A hits the target is p_A, and the probability B hits is p_B. The first to hit the target wins. What is the probability A wins?

Solution. Denote F as the event that A hits the target on the first try. We have

$$\begin{aligned} P(A \text{ wins}) &= P(A \text{ wins}|F)P(F) + P(A \text{ wins}|F^c)P(F^c) \\ &= 1 \cdot p_A + (1 - p_B)P(A \text{ wins}) \cdot (1 - p_A) \end{aligned}$$

Solving for $P(A \text{ wins})$ yields

$$P(A \text{ wins}) = \frac{p_A}{1 - (1 - p_A)(1 - p_B)}$$

□

4. Give an example of 4 dependent random events $A_1, \cdots, A_4$ such that any 3 of them are independent.

Solution. Let $S = \{(x_1, \cdots, x_4) \in \{0,1\}^4 : x_1 + \cdots x_4 \text{ is even}\}$. We can enumerate this set to be

$$\begin{matrix} (0,0,0,0) & (0,0,1,1) & (0,1,0,1) & (0,1,1,0) \\ (1,0,0,1) & (1,0,1,0) & (1,1,0,0) & (1,1,1,1) \end{matrix}$$

Assign a probability of 1/8 to each of these events. Now denote

$$A_i = \{(x_1, \cdots, x_4) \in S : x_i = 0\}$$

We can calculate, for all i, j, k all different, that

$$P(A_i A_j A_k) = P(A_i)P(A_j)P(A_k) = \frac{1}{8}$$

But,

$$P(A_1 A_2 A_3 A_4) = \frac{1}{8} \neq \frac{1}{16} = P(A_1)P(A_2)P(A_3)P(A_4)$$

□

5. Let $A_1, \cdots, A_{2000} \subseteq A$ with $|A_i| \geq 6$ and at least one pair i, j such that $A_i \neq A_j$ (that is, not all A_i are the same). Prove that there are more than 100 distinct partitions of A into 5 disjoint subsets $E_1, \cdots, E_5$ such that each A_i contains elements that belong to at least two of the E_i.

Solution. The strategy here is certainly clever. We know that

$$P(C) = \frac{|C|}{|N|}$$

where

$$\begin{aligned} C &= \{\{E_1, \cdots, E_5\} : \text{Each } A_i \text{ contains elements that belong to at least two } E_i\} \\ N &= \text{All possible partitions of } A \text{ into five sets} \end{aligned}$$

We want to prove that $|C| > 100$, and hence suffices to prove that $P(C) \cdot |N| > 100$. This is interesting because we would think that calculating $P(C)$ would involve calculating $|C|$ anyway, and hence the reasoning is circular. But, through the approach that you will see, this will bypass calculating $|C|$ by approximating it with a lower bound.

First, we can express $C = (\bigcup_{i=1}^{2000} B_i)^c$, where

$$B_i = \{\{E_1, \cdots, E_5\} : A_i \text{ contains elements that belong to at most one } E_i\}$$

We can calculate $P(B_i) = \frac{5}{5^{|A_i|}} = \frac{1}{5^{|A_i|-1}}$, since the probability of getting all elements of A_i into one E_i is $\frac{1}{5^{|A_i|}}$, but then multiply by the 5 choices of E_i we have. Hence,

$$\begin{aligned} P\left(\bigcup_{i=1}^{2000} B_i\right) &\leq \sum_{i=1}^{2000} P(B_i) && \text{Boole's inequality} \\ &= \sum_{i=1}^{2000} \frac{1}{5^{|A_i|-1}} \\ &\leq \sum_{i=1}^{2000} \frac{1}{5^5} && \text{Since } |A_i| \geq 6 \\ &= \frac{2000}{5^5} < \frac{2}{3} \end{aligned}$$

Hence, $P(C) = 1 - P\left(\bigcup_{i=1}^{2000} B_i\right) > \frac{1}{3}$. Next, using the condition that at least A_i, A_j are different, this implies that $|A| \geq 7$. Hence $|N|$ is the number of ways to partition at least 7 elements into at most 5 sets. For those familiar with Stirling numbers of the second kind, this would be

$$|N| = \sum_{k=1}^{5} S_2(|A|, k) \geq \sum_{k=1}^{5} S_2(7, k) = 1 + 63 + 301 + 350 + 140 = 855$$

For those not familiar, we could use another lower bound

$$|N| \geq \frac{5^7}{5!} \approx 651.04$$

This bound is derived from the fact that, if we imagine the $E_1, \cdots, E_5$ to be labeled sets, then clearly the number of ways to put elements into these sets is $5^{|A|} \geq 5^7$. If they were unlabeled, we would correct the over-counting by dividing out the number of ways to order these sets, namely $5! = 120$. But, dividing by 120 actually now undercounts everything. The reason why is, say I partition $\{1, \cdots, 7\}$ into $\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{\}, \{\}$. Since two of these E_i were not chosen, and were already indistinguishable, we should only divide by $5 \cdot 4 \cdot 3 = 60$ here instead of uniformly dividing by 120. This is where the under-estimate comes from, and where we derived the lower bound. Hence,

$$|C| = P(C) \cdot |N| > \frac{1}{3} \cdot 651 = 217 > 100$$

Clearly, the problem could've asked for the bound of 217 (or even larger if we counted $|N|$ exactly) instead of 100. $\square$

6. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if it satisfies

- $d(x_1, x_2) \geq 0$ with equality if and only if $x_1 = x_2$
- $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$
- $d(x_1, x_2) \leq d(x_1, x_3) + d(x_2, x_3)$ for all $x_1, x_2, x_3 \in X$

Define $d_1, d_2 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ as

$$d_1(A, B) = P(A \Delta B)$$
$$d_2(A, B) = \begin{cases} \frac{P(A\Delta B)}{P(A\cup B)}, & \text{if } P(A \cup B) \neq 0 \\ 0, & \text{if } P(A \cup B) = 0 \end{cases}$$

where $A\Delta B = (A \cup B) - (A \cap B) = (A^c \cap B) \cup (A \cap B^c)$ (symmetric difference). Prove that d_1, d_2 are valid metrics.

Solution. This problem is far harder than anything you will see on an exam or homework in BIO230, and perhaps even BIO240. We will need two lemmas, both of which are not trivial:

Lemma 1 $A\Delta B \subseteq (A\Delta C) \cup (B\Delta C)$

Proof. First, one would need to verify Δ is associative and commutative. Commutativity is not hard, and associativity is tedious, but doesn't involve any tricks. Then we would have the identity $(A\Delta B) = (A\Delta C)\Delta(B\Delta C)$. Since

$$(A\Delta C)\Delta(B\Delta C) \subseteq (A\Delta C) \cup (B\Delta C)$$

the lemma follows. □

Lemma 2 $|P(A \cap C) - P(B \cap C)| \leq P(A\Delta B)$

Proof. Let $X = A \cap C$ and $Y = B \cap C$. Then

$$\begin{aligned} |P(X) - P(Y)| &= |(P(X - Y) + P(X \cap Y)) - (P(Y - X) + P(X \cap Y))| \\ &= |P(X - Y) - P(Y - X)| \\ &\leq P(X - Y) + P(Y - X) \qquad \text{Triangle inequality} \\ &= P((X - Y) \cup (Y - X)) \qquad \text{Disjointness} \\ &= P(C \cap [(A - B) \cup (B - A)]) \end{aligned}$$

$$= P(C \cap [A\Delta B])$$
$$\leq P(A\Delta B)$$

$\square$

Now verify the metric conditions for d_1:

- $d_1(A, B) = P(A\Delta B) \geq 0$, with $P(A\Delta A) = P(\emptyset) = 0$.
- $d_1(A, B) = P(A\Delta B) = P(B\Delta A) = d_1(B, A)$.
- $d_1(A, B) = P(A\Delta B) \leq P((A\Delta C) \cup (B\Delta C)) \leq P(A\Delta C) + P(B\Delta C) = d_1(A, C) + d_1(B, C)$, where we applied Lemma 1 and Boole's inequality

For d_2, the case when $P(A \cup B) = 0$ is straightforward to consider, so assume that we are working with non-null sets. We have

- Verify same way as d_1
- Verify same way as d_1
- Note that we can express

  $$d_2(A, B) = \frac{d_1(A, B)}{P(A \cap B) + d_1(A, B)}$$

  Consider the function $f(x) = \frac{x}{c+x}$ for fixed constant $c > 0$. Clearly for $x \geq 0$, $f(x)$ is a monotone increasing function; hence $x \leq y \implies f(x) \leq f(y)$. Let $x = d_1(A, B), y = d_1(A, C) + d_1(B, C), c = P(A \cap B)$ to get

  $$\begin{aligned} d_2(A, B) &= \frac{d_1(A, B)}{P(A \cap B) + d_1(A, B)} \\ &\leq \frac{d_1(A, C) + d_1(B, C)}{P(A \cap B) + d_1(A, C) + d_1(B, C)} \\ &= \frac{d_1(A, C)}{P(A \cap B) + d_1(A, C) + d_1(B, C)} + \frac{d_1(B, C)}{P(A \cap B) + d_1(A, C) + d_1(B, C)} \end{aligned}$$

  It suffices to show that

  $$P(A \cap B) + d_1(A, C) + d_1(B, C) \geq P(A \cup C)$$

  Or equivalently,

  $$P(A \cap B) + d_1(B, C) \geq P(A \cap C)$$

Or equivalently,

$$P(B\Delta C) \geq P(A \cap C) - P(A \cap B)$$

which is true by Lemma 2. Hence,

$$\begin{aligned} d_2(A,B) &\leq \frac{d_1(A,C)}{P(A \cap B) + d_1(A,C) + d_1(B,C)} + \frac{d_1(B,C)}{P(A \cap B) + d_1(A,C) + d_1(B,C)} \\ &\leq \frac{d_1(A,C)}{P(A \cup C)} + \frac{d_1(B,C)}{P(B \cup C)} \\ &= d_2(A,C) + d_2(B,C) \end{aligned}$$

□