1. Let $f : [0, \infty) \to \mathbb{R}$ be a bounded, continuous function. Compute

$$\lim_{n\to\infty} 2^n \int_0^\infty \cdots \int_0^\infty f\left(\frac{x_1 + \cdots + x_n}{n}\right) e^{-2(x_1+\cdots+x_n)} dx_1 \cdots dx_n$$

[Hint: Apply the strong law of large numbers to a suitably chosen iid sequence.]

Solution. Let $X_1, \cdots, X_n \overset{\text{iid}}{\sim} \text{Exp}(2)$. The value we are trying to compute is

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[f(\overline{X}_n)] &= \mathbb{E}\left[\lim_{n\to\infty} f(\overline{X}_n)\right] && \text{by boundedness} \\ &= \mathbb{E}\left[f\left(\lim_{n\to\infty} \overline{X}_n\right)\right] && \text{by continuity}\end{aligned}$$

By SLLN, $\overline{X}_n \overset{\text{a.s.}}{\to} \mathbb{E}X_1 = \frac{1}{2}$, hence $f(\overline{X}_n) \overset{\text{a.s.}}{\to} f(\frac{1}{2})$, hence $\mathbb{E}[f(\overline{X}_n)] \to f(\frac{1}{2})$. $\square$

2. Suppose $\{X_k\}$ independent with

$$\mathbb{P}(X_k = x) = \begin{cases} 1 - \frac{1}{k\log(1+k)} & x = 0 \\ \frac{1}{2k\log(1+k)} & x = -k, k \\ 0 & \text{otherwise} \end{cases}$$

Show that $\overline{X}_n \xrightarrow{\mathcal{P}} 0$, but $\overline{X}_n \overset{\text{a.s.}}{\not\to} 0$.

Solution. Note that $\mathbb{E}X_k = 0$ and $\mathbb{E}X_k^2 = \frac{k}{\log(1+k)}$; this implies that $\mathbb{E}X_k^2$ is monotonically increasing in k. We have

$$\mathbb{P}(|\overline{X}_n| > \varepsilon) \le \frac{\mathbb{E}\overline{X}_n^2}{\varepsilon^2} = \frac{1}{\varepsilon^2}\frac{1}{n^2}\sum_{k=1}^{n}\mathbb{E}X_k^2 \le \frac{1}{\varepsilon^2}\frac{1}{n}\mathbb{E}X_n^2 = \frac{1}{\varepsilon^2}\frac{1}{\log(1+n)} \to 0$$

Showing $\overline{X}_n \xrightarrow{\mathcal{P}} 0$. In showing non a.s. convergence, we just have to show it for *some* ε, so let's let $\varepsilon = 1$ (we will shortly see why this choice is appropriate). Note that

$$\{|\overline{X}_n| \ge 1\} \supseteq \{\overline{X}_n \ge 1\} = \{S_n \ge n\} \supseteq \{S_{n-1} \ge 0, X_n = n\}$$

Hence

$$\begin{aligned} \sum_{n=1}^{\infty}\mathbb{P}(|\overline{X}_n| \ge 1) &\ge \sum_{n=1}^{\infty}\mathbb{P}(S_{n-1} \ge 0)\mathbb{P}(X_n = n) \\ &\ge \sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(X_n = n) \\ &= \sum_{n=1}^{\infty}\frac{1}{4n\log(1+n)} \end{aligned}$$

By the integral test, the last sum is ∞, hence $\sum_{n=1}^{\infty}\mathbb{P}(|\overline{X}_n| \ge 1) = \infty$, so $\overline{X}_n \overset{\text{a.s.}}{\not\to} 0$. □

3. Show that Lindeberg's condition is satisfied when $\{X_k\}$ iid with mean μ and variance σ^2.

Proof. We have

$$\frac{1}{D_n^2}\sum_{k=1}^{n}\int_{\{x:|x-\mu|\geq\varepsilon D_n\}}(x-\mu)^2dF(x)=\frac{n}{n\sigma^2}\int_{\{x:|x-\mu|\geq\varepsilon D_n\}}(x-\mu)^2dF(x)$$

Since $\{x:|x-\mu|\geq\varepsilon D_n\}\to\emptyset$, the entire expression goes to 0. □

4. Let X, Y independent with $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$. Show that

$$\frac{(X-\lambda)-(Y-\mu)}{\sqrt{X+Y}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as } \lambda, \mu \to \infty$$

[Hint: Rewrite $\frac{1}{\sqrt{X+Y}} = \frac{1}{\sqrt{\lambda+\mu}}\frac{\sqrt{\lambda+\mu}}{\sqrt{X+Y}}$ and apply Slutsky's.]

Proof. Let $Z = \frac{(X-\lambda)-(Y-\mu)}{\sqrt{\lambda+\mu}}$. We have

$$M_Z(t) = \exp\left(\lambda(e^{t/\sqrt{\lambda+\mu}} - 1) - t\frac{\lambda}{\sqrt{\lambda+\mu}}\right)\exp\left(\mu(e^{-t/\sqrt{\lambda+\mu}} - 1) + t\frac{\mu}{\sqrt{\lambda+\mu}}\right)$$

So

$$\begin{aligned}
\log M_Z(t) &= \left\{\lambda\left(t\frac{1}{\sqrt{\lambda+\mu}} + \frac{t^2}{2}\frac{1}{\lambda+\mu} + O\left(\frac{1}{(\lambda+\mu)^{3/2}}\right)\right) - t\frac{\lambda}{\sqrt{\lambda+\mu}}\right\} \\
&\quad + \left\{\mu\left(-t\frac{1}{\sqrt{\lambda+\mu}} + \frac{t^2}{2}\frac{1}{\lambda+\mu} + O\left(\frac{1}{(\lambda+\mu)^{3/2}}\right)\right) + t\frac{\mu}{\sqrt{\lambda+\mu}}\right\} \\
&= \frac{t^2}{2} + O\left(\frac{\lambda}{(\lambda+\mu)^{3/2}}\right) + O\left(\frac{\mu}{(\lambda+\mu)^{3/2}}\right) \\
&\to \frac{t^2}{2} \quad \text{as } \lambda, \mu \to \infty
\end{aligned}$$

Hence $Z \xrightarrow{\mathcal{D}} N(0,1)$. Next, we show $\frac{X+Y}{\lambda+\mu} \xrightarrow{\mathcal{P}} 1$, which implies that $W \overset{\text{def}}{=} \frac{\sqrt{\lambda+\mu}}{\sqrt{X+Y}} \xrightarrow{\mathcal{P}} 1$ by the continuous mapping theorem. We have

$$\mathbb{P}\left(\left|\frac{X+Y}{\lambda+\mu} - 1\right| \geq \varepsilon\right) \leq \frac{\text{Var}(\frac{X+Y}{\lambda+\mu})}{\varepsilon^2} = \frac{1}{(\lambda+\mu)\varepsilon^2} \to 0$$

Hence, by Slutsky's, $ZW \xrightarrow{\mathcal{D}} N(0,1)$. □