1. The Hydra is a mythical beast from Greek mythology. Suppose the Hydra starts with 3 heads. Every time the hero Hercules swings his sword, he cuts off all existing heads. When a head is cut it has a 1/3 probability of growing back into 3 new heads, independently of the past and all other heads. What is the probability that Hercules defeats Hydra?

   Solution. Each swing can be viewed as a generation. The expected number of heads grown back by each head chopped off on each swing is $\mu = 3 \cdot \frac{1}{3} = 1$. Since the Hydra isn't trivially growing back exactly 1 head every generation, the criticality theorem states that Hercules will defeat Hydra with probability 1. □

2. A gram of uranium contains $X_0 = 2.5 \times 10^{21}$ atoms. Each atom decays into lead after an independent exponentially distributed amount of time with rate λ. Let X_t be the number of uranium atoms remaining after t years.

   (a) The half-life of uranium is 4.5×10^9 years. Use this to find λ

   (b) Show that $X_t \sim \text{Binom}(X_0, p_t)$, and find p_t.

   (c) Let $\tau_1 = \{t : X_t = X_0 - 1\}$ be the time until the first atom decays into lead. Find μ_{τ_1} and σ_{τ_1}.

   (d) Estimate the mean and standard deviation of the number of atoms that decay during the first second.

   (e) In general, let $\tau_k = \{t : X_t = X_0 - k\}$ be the time until k atoms have decayed into lead. Then, $\tau_{(1-p)X_0-1}$ is the time until a proportion p of uranium atoms remain. Using the approximation $\sum_{k=1}^m \frac{1}{k} \approx \log(m) + \gamma + \frac{1}{2m}$ (where $\gamma \approx 0.5772$), show that

$$\mathbb{E}\tau_{(1-p)X_0-1} \approx -\frac{\log p}{\lambda}$$

   which is the formula for exponential decay. Similarly, using the approximation $\sum_{k=1}^m \frac{1}{k^2} \approx \frac{\pi^2}{6} - \frac{1}{m}$, show that $\text{Var}(\tau_{(1-p)X_0-1}) \approx 0$ for large X_0.

Solution.

(a) The half life can be said to be the median time until which an atom decays. So,

$$\frac{1}{2} = \Pr(T > t_{0.5}) = e^{-\lambda t_{0.5}} \implies \lambda = \frac{\log(2)}{t_{0.5}} = 1.54 \times 10^{-10} \text{ yr}^{-1}$$

(b) Let T_k represent the time of survival for atom k. We can represent $X_t = \sum_{k=1}^{X_0} \mathbb{I}(T_k > t)$. Since $\mathbb{I}(T_k > t) \sim \text{Ber}(e^{-\lambda t})$, and sum of Bernoulli random variables is binomial, we have $X_t \sim \text{Binom}(X_0, e^{-\lambda t})$.

(c) Realize that $\tau_1 = \min(T_1, \cdots, T_{X_0})$. Since each $T_k \sim \text{Exp}(\lambda)$, and minimum of exponentials is exponential, we have $\tau_1 \sim \text{Exp}(\lambda X_0)$. So,

$$\mu_{\tau_1} = \sigma_{\tau_1} = \frac{1}{X_0 \lambda} = 2.597 \times 10^{-12} \text{ yr} = 8.19 \times 10^{-5} \text{ sec}$$

(d) Realize that $N_t = X_0 - X_t$, with $t = 3.17 \times 10^{-8}$ year (or 1 second), is our desired random variable. This follows $\text{Binom}(X_0, 1 - e^{-\lambda t})$. Also, as an approximation, we can imagine that in one second, the number of atoms remaining does not change much, so N_t is approximately $\text{Pois}(\lambda X_0 t)$. The expected value under both schemes, respectively, is $X_0(1-e^{-\lambda t})$ and $\lambda X_0 t$. Both formulas effectively produce the same result with $\mathbb{E}N_t \approx 1.22 \times 10^4$ and $\mathrm{sd}(N_t) \approx 110.5$.

(e) Denote $t_k = \tau_{k+1} - \tau_k$, with $\tau_0 = 0$. That is, t_k represents the time between the kth and $(k+1)$th atom decay. We see that $t_k \sim \mathrm{Exp}((X_0 - k)\lambda)$. Then, $\tau_n = t_0 + \cdots + t_n$ is the time until the $(n+1)$th atom decay. Taking expectation,

$$\begin{aligned}
\mathbb{E}\tau_{(1-p)X_0-1} &= \sum_{k=0}^{(1-p)X_0-1} \mathbb{E}t_k \\
&= \sum_{k=0}^{(1-p)X_0-1} \frac{1}{(X_0-k)\lambda} && \text{Since } t_k \sim \mathrm{Exp}((X_0-k)\lambda) \\
&= \sum_{k=pX_0+1}^{X_0} \frac{1}{k\lambda} && \text{Re-index } k \mapsto X_0 - k \\
&= \frac{1}{\lambda}\left(\sum_{k=1}^{X_0}\frac{1}{k} - \sum_{k=1}^{pX_0}\frac{1}{k}\right)
\end{aligned}$$

Using the approximation $\sum_{k=1}^{n} \frac{1}{k} \approx \log(n) + \gamma + \frac{1}{2n}$, we have

$$\begin{aligned}
\mathbb{E}\tau_{(1-p)X_0-1} &\approx \frac{1}{\lambda}\left(\log(X_0) + \gamma + \frac{1}{2X_0} - \log(pX_0) - \gamma - \frac{1}{2pX_0}\right) \\
&= -\frac{\log(p)}{\lambda} + \frac{1}{2\lambda X_0}\left(1 - \frac{1}{p}\right)
\end{aligned}$$

Since X_0 is extremely large, and assuming p is not too close to 0, the second term is essentially 0, and we arrive at our answer. For the variance,

$$\begin{aligned}
\mathrm{Var}(\tau_{(1-p)X_0-1}) &= \sum_{k=0}^{(1-p)X_0-1} \frac{1}{(X_0-k)^2\lambda^2} \\
&= \frac{1}{\lambda^2}\left(\sum_{k=1}^{X_0}\frac{1}{k^2} - \sum_{k=1}^{pX_0}\frac{1}{k^2}\right) \\
&\approx \frac{1}{\lambda^2}\left(\frac{\pi^2}{6} - \frac{1}{X_0} - \frac{\pi^2}{6} + \frac{1}{pX_0}\right) \\
&= \frac{1}{\lambda^2 X_0}\left(\frac{1}{p} - 1\right)
\end{aligned}$$

For large X_0, this expression is nearly 0.

□

3. Gas stations are distributed along a Nevada desert highway according to a Poisson process with rate 1/(50 miles). Your car has a 10 gallon gas tank and gets 25 miles per gallon. When your tank drops below g gallons, you stop to refill it at the next station you pass.

   (a) If $g = 4$ and you start with a full tank at Lake Tahoe, what is the chance you run out of gas before reaching the Utah state line, 400 miles to the east?

   (b) Same question with $g = 6$.

Solution.

(a) At $g = 4$, we will use 6 gallons and travel $6 \times 25 = 150$ miles before even considering to look for a gas station. Then, the car has the capacity of only traveling $4 \times 25 = 100$ more miles before running out of gas. The probability of having no gas stations in that interval is $\mathbb{P}(N_{150+100} - N_{150} = 0) = e^{-2}$, where we used the fact that $N_{250} - N_{150} \sim \text{Pois}(100 \times \frac{1}{50})$. If we do fill up our tank in the interval [150, 250), then no matter what, we will be able to reach the Utah state line, since our tank has the capacity to travel $10 \times 25 = 250$ miles, which is more than the distance left to travel. So, the probability we won't reach our destination is e^{-2}.

(b) At $g = 6$, we will travel $4 \times 25 = 100$ miles before seeking a gas station. Then, we seek to find a gas station in the interval [100, 250) before we run out of gas. The probability we don't find a gas station in this interval is $\mathbb{P}(N_{250} - N_{100} = 0) = e^{-150 \times \frac{1}{50}} = e^{-3}$. However, another possibility can occur: we find a gas station at $t \in [100, 150)$, begin seeking for a gas station at $t + 100$, but since our car can only travel 250 miles, we would need to find another gas station in the interval $[t + 100, t + 250)$ to guarantee arriving at our destination. This happens with probability $\mathbb{P}(N_{150} - N_{100} > 0, N_{t+250} - N_{t+100} = 0)$. By independent increments, this is equal to

$$\mathbb{P}(N_{150} - N_{100} > 0)\mathbb{P}(N_{t+250} - N_{t+100} = 0) = (1 - e^{-1})e^{-3}$$

Therefore, the total probability of not making our destination is $e^{-3} + (1 - e^{-1})e^{-3} = 2e^{-3} - e^{-4}$.

□

4. You have coins $C_1, C_2, \cdots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If n coins are tossed, what is the probability that the number of heads is odd?

Solution. Let $C_k = 1$ if heads and 0 if tails, and let $C = \sum_{k=1}^{n} C_k$. Note that

$$g_C(t) = \prod_{k=1}^{n} g_{C_k}(t) = \prod_{k=1}^{n} \left(\frac{2k}{2k+1} + \frac{1}{2k+1} t \right)$$

Also note that

$$\begin{aligned} g_C(1) &= 1 \\ g_C(-1) &= \prod_{k=1}^{n} \left(\frac{2k-1}{2k+1} \right) = \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{2n-1}{2n+1} = \frac{1}{2n+1} \end{aligned}$$

We see that

$$\mathbb{P}(C \text{ odd}) = \frac{g_C(1) - g_C(-1)}{2} = \frac{1 - \frac{1}{2n+1}}{2} = \frac{n}{2n+1}$$

□