Lab 11

Stochastic Processes

11.1. Probability Generating Functions

Definition 11.1.1. If X is a discrete random variable with $\text{supp}(X) \subseteq \mathbb{N}_0$, then the *probability-generating function* of X is defined as

$$g_X(t) \stackrel{\text{def}}{=} \mathbb{E}t^X = \sum_{x=0}^{\infty} p(x)t^x$$

where $p(x) = \Pr(X = x)$, the pmf of X. If $\max(X) = M$, then we can let $p(x) = 0$ for $x > M$. Throughout these notes, we assume all random variables to be DRV which take non-negative integer values. The following is a list of some simple properties of PGF's.

1. $g_X(1) = 1$
2. $p(k) = \frac{g_X^{(k)}(0)}{k!}$
3. $X \stackrel{d}{=} Y \iff g_X(z) = g_Y(t)$
4. $\mathbb{E}X^{\underline{k}} \stackrel{\text{def}}{=} \mathbb{E}[X(X-1)\cdots(X-k+1)] = g_X^{(k)}(1)$, assuming $\mathbb{E}|X|^k < \infty$
5. $g_X(e^t) = M_X(t)$, where $M_X(t)$ is the moment-generating function of X

Proof. We have

1. $g_X(1) = \sum_{x=0}^{\infty} p(x) = \mathbb{P}(X \leq \infty) = 1$
2. $\frac{g_X^{(k)}(0)}{k!} = \{\frac{1}{k!}\sum_{x=k}^{\infty} x^{\underline{k}} p(x) t^{x-k}\}_{z=0} = \frac{k!}{k!}p(k) = p(k)$
3. The forward direction is trivial. For the backwards direction, it follows from property (2) that $g_X(t) = g_Y(t) \implies \frac{g_X^{(k)}(0)}{k!} = \frac{g_Y^{(k)}(0)}{k!} \implies p_X(k) = p_Y(k) \quad \forall k \in \mathbb{N}_0 \implies X \stackrel{\mathcal{D}}{=} Y$
4. $g_X^{(k)}(1) = \{\sum_{x=k}^{\infty} x^{\underline{k}} p(x) t^{x-k}\}_{t=1} = \sum_{x=k}^{\infty} x^{\underline{k}} p(x) = \sum_{x=0}^{\infty} x^{\underline{k}} p(x) = \mathbb{E}X^{\underline{k}}$, since $x^{\underline{k}} = 0$ for $x < k$
5. $g_X(e^t) = \mathbb{E}(e^t)^X = \mathbb{E}e^{tX} = M_X(t)$

□

Example 11.1.1. (Variance formula) The variance of X would be

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \\ &= \mathbb{E}[X(X-1)] + \mathbb{E}X - (\mathbb{E}X)^2 \\ &= g_X''(1) + g_X'(1) - [g_X'(1)]^2 \\ &= g_X''(1)/g_X(1) + g_X'(1)/g_X(1) - [g_X'(1)/g_X(1)]^2 \\ &= \{(\log g_X(t))' + (\log g_X(t))''\}_{t=1}\end{aligned}$$

Example 11.1.2. Let $\{X_i\}$ be a (possibly infinite) sequence of independent (but not necessarily identically distributed) random variables, and let $\{a_i\}$ be a sequence of scalars. Let $S_n = \sum_{i=1}^n a_i X_i$ and N be another discrete random variable with $\text{supp}(N) \subseteq \mathbb{N}_0$. Then,

$$g_{S_N}(t) = \sum_{n=1}^{\infty} \mathbb{P}(N=n) \prod_{i=1}^{n} g_{X_i}(t^{a_i})$$

Proof. Applying the law of total expectation,

$$g_{S_N}(t) = \mathbb{E}t^{S_N} = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}(t^{S_N}|N=n) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}t^{S_n}$$

Analyzing $\mathbb{E}t^{S_n}$,

$$\mathbb{E}t^{S_n} = \mathbb{E}t^{\sum_{i=1}^n a_i X_i} = \mathbb{E}\prod_{i=1}^{n} t^{a_i X_i} = \prod_{i=1}^{n} \mathbb{E}t^{a_i X_i} = \prod_{i=1}^{n} g_{X_i}(t^{a_i})$$

where the third equality is valid because $\{X_i\}$ are independent. Then,

$$g_{S_N}(t) = \sum_{n=1}^{\infty} \mathbb{P}(N=n) \prod_{i=1}^{n} g_{X_i}(t^{a_i})$$

where we subtly omitted the $n = 0$ index because that term is 0. □

Corollary 11.1.1. If $X_1, X_2, \cdots$ were iid, and $\{a_i\}$ was a constant sequence at 1, then

$$g_{S_N}(t) = g_N(g_X(t))$$

Furthermore, if N were a constant random variable $\mathbb{P}(N = n) = 1$, then

$$g_{S_N}(t) = g_{S_n}(t) = (g_X(t))^n$$

Example 11.1.3. (Sicherman dice) You have two pairs of fair dice. For the first pair, both dice (X_1, X_2) have the labeling $\{1, 2, 3, 4, 5, 6\}$. For the second pair, Y_1 has labeling $\{1, 2, 2, 3, 3, 4\}$ and Y_2 has labeling $\{1, 3, 4, 5, 6, 8\}$. Let $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. Prove that $X \stackrel{\mathcal{D}}{=} Y$.

Solution. The pgf for X is

$$\begin{aligned} g_X(t) &= \left(\frac{1}{6}t + \cdots + \frac{1}{6}t^6\right)^2 \\ &= \frac{1}{36}(t^2 + 2t^3 + 3t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12}) \end{aligned}$$

We also have

$$\begin{aligned} g_Y(t) = g_{Y_1}(t)g_{Y_2}(t) &= \frac{1}{6} \cdot (t + 2t^2 + 2t^3 + t^4) \cdot \frac{1}{6} \cdot (t + t^3 + t^4 + t^5 + t^6 + t^8) \\ &= \frac{1}{36}(t^2 + 2t^3 + 3t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12}) \end{aligned}$$

Since $g_X = g_Y$, we have $X \stackrel{\mathcal{D}}{=} Y$. □

11.2. Poisson Processes

Example 11.2.1. Cars and buses arrive at a bridge according to independent Poisson processes at a rate of $\lambda_c = 3 \ \frac{\text{cars}}{\text{min}}$ and $\lambda_b = 1 \ \frac{\text{car}}{\text{min}}$.

(a) What is the chance that the first 2 vehicles to arrive are cars?

(b) What is the chance that exactly 17 vehicles arrive between 12:00 and 12:05?

(c) Given that exactly 17 vehicles arrive between 12:00 and 12:05, what is the chance that all of them are cars?

(d) What is the chance that strictly more buses arrive than cars between 12:00 and 12:01?

(e) It takes 10 seconds for a car to cross the bridge and 15 seconds for a bus to cross the bridge. At any given instance, what is the probability the bridge will be empty?

Solution.

(a) **Slick Way** Suppose we wait until we observe a car or a bus. The probability the car comes first is $\frac{\lambda_c}{\lambda_c+\lambda_b}$. Then, restart the exponential race, and wait for another car or bus to come by. By the memoryless property, this probability is $\frac{\lambda_c}{\lambda_c+\lambda_b}$. So, the probability of observing two cars before the bus is $\left(\frac{\lambda_c}{\lambda_c+\lambda_b}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

Long Way Let T_n^c, T_n^b denote the time until the arrival of the n vehicle for cars and buses, respectively. The problem asks to find $\mathbb{P}(T_2^c < T_1^b)$. We know that $T_2^c \sim \text{Gamma}(2, 3)$ and $T_1^b \sim \text{Exp}(1)$. So,

$$\begin{aligned} \mathbb{P}(T_2^c < T_1^b) &= \int_0^\infty \int_0^{t_b} \lambda_c e^{-\lambda_c t_c}(\lambda_c t_c)\lambda_b e^{-\lambda_b t_b} dt_c dt_b \\ &= \text{(Algebra)} \\ &= \left(\frac{\lambda_c}{\lambda_c + \lambda_b}\right)^2 = \frac{9}{16} \end{aligned}$$

(b) Denote N_t^c, N_t^b denote the number of cars and buses, respectively, that pass in a time t. We have

$$\mathbb{P}(N_5^c + N_5^b = 17) = e^{-20}\frac{20^{17}}{17!} \approx 0.07595$$

(c) We have $\mathbb{P}(N_5^c = 17, N_5^b = 0) = e^{-15}\frac{15^{17}}{17!}e^{-5} \approx 0.0005709$. So,

$$\mathbb{P}(N_5^c = 17, N_5^b = 0 | N_5^c + N_5^b = 17) = \frac{\mathbb{P}(N_5^c = 17, N_5^b = 0)}{\mathbb{P}(N_5^c + N_5^b = 17)} \approx 0.007517.$$

(d) We are looking for $\mathbb{P}(N_1^c < N_1^b)$. Expanding,

$$\begin{aligned}\mathbb{P}(N_1^c < N_1^b) &= \sum_{m=1}^{\infty} \mathbb{P}(N_1^c < m)\mathbb{P}(N_1^b = m) \\ &= \sum_{m=1}^{\infty}\sum_{n=0}^{m-1} \mathbb{P}(N_1^c = n)\mathbb{P}(N_1^b = m) \\ &= \sum_{m=1}^{\infty}\sum_{n=0}^{m-1} e^{-3}\frac{3^n}{n!}e^{-1}\frac{1}{m!} \\ &\approx 0.0939\end{aligned}$$

(e) The probability that the bridge is empty at some t (in minutes) is

$$\mathbb{P}\left(N_t^b - N_{t-\frac{1}{4}}^b = 0, N_t^c - N_{t-\frac{1}{6}}^c = 0\right)$$

Where we converted 15 sec to 1/4 min, 10 sec to 1/6 min. Since $N_t^b - N_{t-\frac{1}{4}}^b \left(t - \frac{1}{4}\right) \sim$ Pois$(\frac{1}{4} \cdot 1)$ and $N_t^c - N_{t-\frac{1}{6}}^c \sim$ Pois$(\frac{1}{6} \cdot 3)$ independently, we have

$$\mathbb{P}\left(N_t^b - N_{t-\frac{1}{4}}^b = 0\right)\mathbb{P}\left(N_t^c - N_{t-\frac{1}{6}}^c = 0\right) = e^{-1/4}e^{-1/2} = e^{-3/4}$$

□

11.3. Additional Problems

1. The Hydra is a mythical beast from Greek mythology. Suppose the Hydra starts with 3 heads. Every time the hero Hercules swings his sword, he cuts off all existing heads. When a head is cut it has a 1/3 probability of growing back into 3 new heads, independently of the past and all other heads. What is the probability that Hercules defeats Hydra?

2. A gram of uranium contains $X_0 = 2.5 \times 10^{21}$ atoms. Each atom decays into lead after an independent exponentially distributed amount of time with rate λ. Let X_t be the number of uranium atoms remaining after t years.

   (a) The half-life of uranium is 4.5×10^9 years. Use this to find λ

   (b) Show that $X_t \sim \text{Binom}(X_0, p_t)$, and find p_t.

   (c) Let $\tau_1 = \{t : X_t = X_0 - 1\}$ be the time until the first atom decays into lead. Find μ_{τ_1} and σ_{τ_1}.

   (d) Estimate the mean and standard deviation of the number of atoms that decay during the first second.

   (e) In general, let $\tau_k = \{t : X_t = X_0 - k\}$ be the time until k atoms have decayed into lead. Then, $\tau_{(1-p)X_0-1}$ is the time until a proportion p of uranium atoms remain. Using the approximation $\sum_{k=1}^{m} \frac{1}{k} \approx \log(m) + \gamma + \frac{1}{2m}$ (where $\gamma \approx 0.5772$), show that

   $$\mathbb{E}\tau_{(1-p)X_0-1} \approx -\frac{\log p}{\lambda}$$

   which is the formula for exponential decay. Similarly, using the approximation $\sum_{k=1}^{m} \frac{1}{k^2} \approx \frac{\pi^2}{6} - \frac{1}{m}$, show that $\text{Var}(\tau_{(1-p)X_0-1}) \approx 0$ for large X_0.

3. Gas stations are distributed along a Nevada desert highway according to a Poisson process with rate 1/(50 miles). Your car has a 10 gallon gas tank and gets 25 miles per gallon. When your tank drops below g gallons, you stop to refill it at the next station you pass.

   (a) If $g = 4$ and you start with a full tank at Lake Tahoe, what is the chance you run out of gas before reaching the Utah state line, 400 miles to the east?

   (b) Same question with $g = 6$.

4. You have coins $C_1, C_2, \cdots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If n coins are tossed, what is the probability that the number of heads is odd?