1. Let X, Y be independent RV's such that X has a continuous distribution. Is it true that $X+Y$ has a continuous distribution?

Solution. Yes. Since X, Y are both $\mathcal{B}$-measurable by assumption, their sum is $\mathcal{B}$-measurable with well-defined CDF function (in this class, you need not state this, but it's here for completeness). It suffices to just show that $\mathbb{P}(X+Y=a)=0$ for any real constant a. Since X is continuous, and hence $\mathbb{P}(X=x)=0$ for any x, we have

$$\begin{aligned}\mathbb{P}(X+Y=a) &= \int_{x+y=a} dF_X(x)dF_Y(y) \\ &= \int_{\mathbb{R}} \mathbb{P}(X=a-y)dF_Y(y) \\ &= \int_{\mathbb{R}} 0 \cdot dF_Y(y) \\ &= 0\end{aligned}$$

□

2. Let X, Y, Z be independent and strictly positive RV's. Define $\xi = \frac{X}{Z}$ and $\eta = \frac{Y}{Z}$. For all $a > 0$, prove that

$$\mathbb{P}(\xi < a, \eta < a) \geq \mathbb{P}(\xi < a)\mathbb{P}(\eta < a)$$

Solution. Without loss of generality, take $a = 1$, for otherwise we would be working with RV aZ instead of Z. We have

$$\mathbb{P}(X < Z, Y < Z) - \mathbb{P}(X < Z)\mathbb{P}(Y < Z) = \int_{\mathbb{R}^2} \varphi(x,y)dF_X(x)dF_Y(y)$$

where $\varphi(x,y) = \mathbb{P}(x < Z, y < Z) - \mathbb{P}(x < Z)\mathbb{P}(y < Z)$. It suffices to prove that $\varphi(x,y) \geq 0$ for all x, y. Indeed, if $x < y$, we have

$$\begin{aligned}\varphi(x,y) &= \mathbb{P}(\max(x,y) < Z) - \mathbb{P}(x < Z)\mathbb{P}(y < Z) \\ &= \mathbb{P}(y < Z) - \mathbb{P}(x < Z)\mathbb{P}(y < Z) \\ &= [1 - \mathbb{P}(x < Z)]\mathbb{P}(y < Z) \geq 0\end{aligned}$$

□

and similarly for $x \geq y$.

3. Suppose $X_1, \cdots, X_n$ satisfy

$$\mathbb{P}(X_i = 0) = 1 - \lambda_i \varepsilon, \qquad \mathbb{P}(X_i = 1) = \lambda_i \varepsilon$$

for small $\varepsilon > 0$ and $\lambda_i > 0$. Prove that

$$\mathbb{P}\left(\sum_{i=1}^n X_i = 1\right) = \left(\sum_{i=1}^n \lambda_i\right)\varepsilon + \mathcal{O}(\varepsilon^2)$$
$$\mathbb{P}\left(\sum_{i=1}^n X_i > 1\right) = \mathcal{O}(\varepsilon^2)$$

where $f(x) = \mathcal{O}(g(x))$ if and only if $\left|\frac{f(x)}{g(x)}\right| \leq M$ for some $M > 0$ and all $x \geq x_0$ for some $x_0 > 0$.

Solution. We have

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^n X_i = 1\right) &= \sum_{i=1}^n \mathbb{P}\left(X_i = 1, \{X_j = 0 : j \neq i\}\right) \\
&= \sum_{i=1}^n \left\{\lambda_i \varepsilon \prod_{j \neq i}(1 - \lambda_i \varepsilon)\right\} \\
&= \sum_{i=1}^n \lambda_i \varepsilon \left(1 - \varepsilon \sum_{j \neq i} \lambda_i + \varepsilon^2 \sum_{j_1, j_2 \neq i} \lambda_{j_1}\lambda_{j_2} - \cdots\right) \\
&= \sum_{i=1}^n \lambda_i \varepsilon \left(1 + \mathcal{O}(\varepsilon)\right) \\
&= \left(\sum_{i=1}^n \lambda_i\right)\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

Also,

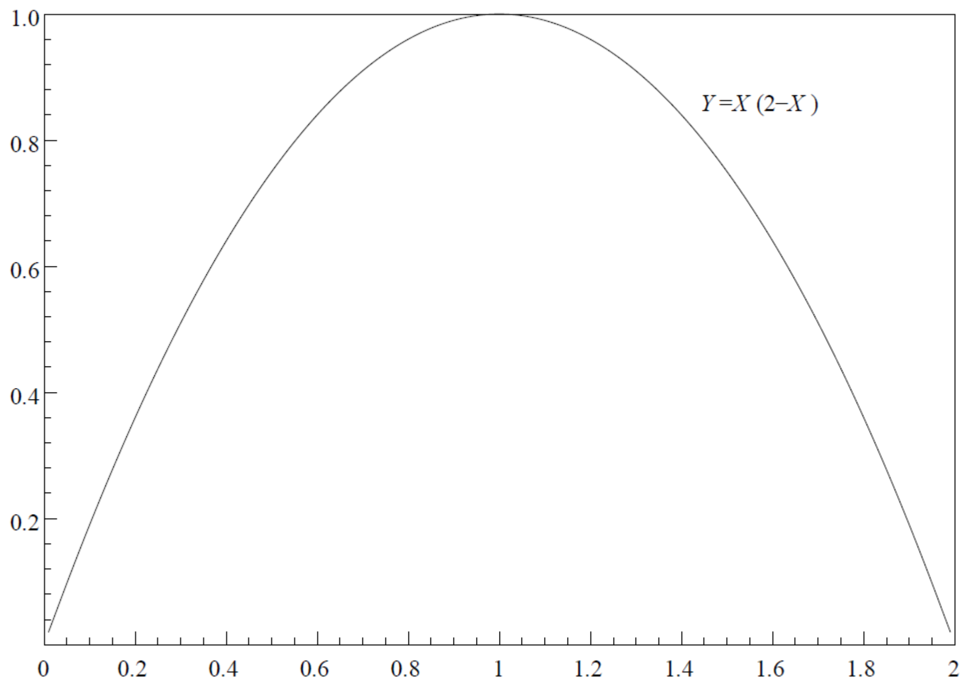
$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^n X_i > 1\right) &= 1 - \mathbb{P}\left(\sum_{i=1}^n X_i = 0\right) - \mathbb{P}\left(\sum_{i=1}^n X_i = 1\right) \\
&= 1 - \prod_{i=1}^n (1 - \lambda_i \varepsilon) - \left\{\left(\sum_{i=1}^n \lambda_i\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right\} \\
&= 1 - \left(1 - \left(\sum_{i=1}^n \lambda_i\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right) - \left(\sum_{i=1}^n \lambda_i\right)\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{O}(\varepsilon^2)
\end{aligned}$$

where we used the fact that $-\mathcal{O}(f(x)) = \mathcal{O}(f(x))$. $\square$

4. Suppose CRV X has PDF $f_X(x) = \frac{1}{2}$ for $0 \leq x \leq 2$. Find the PDF of $Y = X(2 - X)$.

Solution. I will employ the CDF approach from the lab

Step 1: Draw another picture:



We see that

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(X(2-X) \leq y) \\
&= \mathbb{P}((X-1)^2 \geq 1-y) \\
&= \mathbb{P}\left(\left\{X - 1 \geq \sqrt{1-y}\right\} \cup \left\{X - 1 \leq -\sqrt{1-y}\right\}\right) \\
&= \mathbb{P}(X - 1 \geq \sqrt{1-y}) + \mathbb{P}(X - 1 \leq -\sqrt{1-y}) \\
&= \mathbb{P}(X \geq 1 + \sqrt{1-y}) + \mathbb{P}(X \leq 1 - \sqrt{1-y}) \\
&= 1 - F_X(1 + \sqrt{1-y}) + F_X(1 - \sqrt{1-y})
\end{aligned}$$

Step 2: We have

$$\begin{aligned}
f_Y(y) &= F'_Y(y) \\
&= f_X(1 + \sqrt{1-y}) \cdot \frac{1}{2\sqrt{1-y}} + f_X(1 - \sqrt{1-y}) \cdot \frac{1}{2\sqrt{1-y}} \\
&= \frac{1}{2\sqrt{1-y}} \qquad \text{for } 0 \leq y \leq 1
\end{aligned}$$

□