1. (a) Let $M_X(t)$ be the MGF of X, and define $K(t) = \log(M_X(t))$. Show that $K'(0) = \mathbb{E}X$ and $K''(0) = \text{var}(X)$.

   (b) Let

$$M_X(t) = \frac{(2\ell-1)!!}{(2\ell)!!} \frac{(t+2)(t+4)(t+6)\cdots(t+2\ell)}{(t+1)(t+3)(t+5)\cdots(t+2\ell-1)}$$

where we define

$$r!! = \begin{cases} r(r-2)\cdots 5\cdot 3\cdot 1, & r \text{ odd} \\ r(r-2)\cdots 6\cdot 4\cdot 2, & r \text{ even} \end{cases}$$

Show that M_X is a valid MGF (that is, $M_X(0) = 1$). Then, find an approximation for $\mathbb{E}X$ and $\text{var}(X)$ in terms of ℓ. The following information may be helpful:

$$\sum_{k=1}^{m} \frac{1}{k} \doteq \log(m) + \gamma + \frac{1}{2m} \qquad \text{and} \qquad \sum_{k=1}^{m} \frac{1}{k^2} \doteq \frac{\pi^2}{6} - \frac{1}{m}$$

where $\gamma \doteq 0.5772$, known as the Euler-Mascheroni constant.

Solution.

(a) We have

$$K'(0) = \frac{M'(0)}{M(0)} = \mathbb{E}X$$
$$K''(0) = \frac{M''(0)M(0) - (M'(0))^2}{(M(0))^2} = \mathbb{E}X^2 + (\mathbb{E}X)^2 = \text{var}(X)$$

(b) First,

$$K(t) = C + \sum_{n=1}^{\ell} \log(t+2n) - \sum_{n=1}^{\ell} \log(t+2n-1)$$
$$K'(t) = \sum_{n=1}^{\ell} \frac{1}{t+2n} - \sum_{n=1}^{\ell} \frac{1}{t+2n-1}$$
$$K''(t) = -\sum_{n=1}^{\ell} \frac{1}{(t+2n)^2} + \sum_{n=1}^{\ell} \frac{1}{(t+2n-1)^2}$$

where $C = \log((2\ell-1)!!/(2\ell)!!)$. Hence,

$$\mathbb{E}X = \sum_{n=1}^{\ell} \frac{1}{2n} - \sum_{n=1}^{\ell} \frac{1}{2n-1}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{\ell}\frac{1}{n} - \left(\sum_{k=1}^{2\ell-1}\frac{1}{k} - \frac{1}{2}\sum_{k=1}^{\ell}\frac{1}{k}\right) \\
&\doteq \log(\ell) + \gamma + \frac{1}{2\ell} - \left(\log(2\ell-1) + \gamma + \frac{1}{2(2\ell-1)}\right) \\
&= \log\left(\frac{\ell}{2\ell-1}\right) + \frac{\ell-1}{2\ell(2\ell-1)}
\end{aligned}
$$

Letting $\ell \to \infty$, we see that $\mathbb{E}X \to \log(1/2) = -\log(2)$. Also,

$$
\begin{aligned}
\mathrm{var}(X) &= -\sum_{n=1}^{\ell}\frac{1}{(2n)^2} + \sum_{n=1}^{\ell}\frac{1}{(2n-1)^2} \\
&= -\frac{1}{4}\sum_{n=1}^{\ell}\frac{1}{n^2} + \left(\sum_{k=1}^{2\ell-1}\frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{\ell}\frac{1}{k^2}\right) \\
&\doteq -\frac{1}{2}\left(\frac{\pi^2}{6} - \frac{1}{\ell}\right) + \left(\frac{\pi^2}{6} - \frac{1}{2\ell-1}\right) \\
&= \frac{\pi^2}{12} - \frac{1}{2\ell(2\ell-1)}
\end{aligned}
$$

Letting $\ell \to \infty$, we see that $\mathrm{var}(X) \to \pi^2/12$.

□

2. Let X be any non-negative RV. Prove that for any $r > 1$ we have

$$\int_0^\infty \frac{\mathbb{E}(X \wedge y^r)}{y^r} dy = \frac{r}{r-1}\mathbb{E}X^{1/r}$$

where $a \wedge b = \min(a, b)$.

Solution. We have

$$\begin{aligned}
\int_0^\infty \frac{\mathbb{E}(X \wedge y^r)}{y^r} dy &= \int_0^\infty \int_0^\infty \frac{x \wedge y^r}{y^r} dF_X(x) dy && \text{Definition} \\
&= \int_0^\infty \int_0^\infty \frac{x \wedge y^r}{y^r} dy dF_X(x) && \text{Fubini's theorem} \\
&= \int_0^\infty \int_0^{x^{1/r}} \frac{x \wedge y^r}{y^r} dy dF_X(x) + \int_0^\infty \int_{x^{1/r}}^\infty \frac{x \wedge y^r}{y^r} dy dF_X(x) && \text{Partition integration region} \\
&= \int_0^\infty \int_0^{x^{1/r}} \frac{y^r}{y^r} dy dF_X(x) + \int_0^\infty \int_{x^{1/r}}^\infty \frac{x}{y^r} dy dF_X(x) && \text{Definition} \\
&= \int_0^\infty x^{1/r} dy dF_X(x) + \int_0^\infty x \left(\frac{1}{r-1} x^{\frac{-r+1}{r}} \right) dy dF_X(x) && \text{Math} \\
&= \mathbb{E}X^{1/r} + \frac{1}{r-1}\mathbb{E}X^{1/r} && \text{Definition} \\
&= \frac{r}{r-1}\mathbb{E}X^{1/r}
\end{aligned}$$

□

3. Suppose $X_1, \cdots, X_{100} \sim F_X$, where F_X is continuous and strictly increasing. Let $\xi_1 = X_1$ and $\xi_k = X_k \wedge \xi_{k-1}$, where $a \wedge b = \min(a, b)$. Find the expected number of distinct elements in the set $\{\xi_1, \cdots, \xi_{100}\}$.

Remark: There is a nice real-world interpretation to this problem. Suppose drivers $1, \cdots, 100$ are on a single-lane road, with driver 1 in the front, and let X_i by their desired speed to drive at. However, this desired speed may not be realized, since the driver in front might be driving slower, and hence our realized speed is in fact $\xi_k = X_k \wedge \xi_{k-1}$. Then, the number of distinct elements in $\{\xi_1, \cdots, \xi_{100}\}$ correspond to the number of pockets or groups of vehicles traveling together.

Solution. This problem is meant to exemplify the *indicator method* in finding expectations. That is, let D be the number of distinct elements. Then, I claim we can represent D as

$$D = \sum_{k=1}^{100} \mathbb{I}(\xi_k = X_k)$$

where $\mathbb{I}(A)$ is the indicator function, taking 1 if the event A happens, otherwise 0. First, realize that, since each X_k is a CRV, the probability $X_k = \xi_{k-1}$ is 0. Hence, each ξ_k either takes on the value X_k or ξ_{k-1}. If $\xi_k = \xi_{k-1}$, then we do not have a distinct element, but if $\xi_k = X_k$, we do have a distinct element. This information is exactly encoded within each indicator function. Also, note that

$$\mathbb{E}\mathbb{I}(A) = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^c) = \mathbb{P}(A)$$

That is, the expected value of an indicator random variable is simply the probability of the event it's indicating. Now,

$$\mathbb{E}D = \sum_{k=1}^{100} \mathbb{E}\mathbb{I}(\xi_k = X_k) = \sum_{k=1}^{100} \mathbb{P}(\xi_k = X_k) = \sum_{k=1}^{100} \mathbb{P}(\min(X_1, \cdots, X_k) = X_k)$$

since one may iterate the definition of ξ_k to get $\min(X_1, \cdots, X_k)$. Since $X_1, \cdots, X_k$ are continuous, note that $\mathbb{P}(\min(X_1, \cdots, X_k) = X_k)$ is equivalent to saying, if we were to randomly permute $X_1, \cdots, X_k$, what is the probability X_k is first? This is clearly just $\frac{1}{k}$, hence

$$\mathbb{E}D = \sum_{k=1}^{100} \frac{1}{k}$$

□

4. We denote $t^+ = \max(t, 0)$ and $t^- = \max(-t, 0)$. Let X, Y be random variables such that $\mathbb{E}X^+/\mathbb{E}X^- \geq z$ and $\mathbb{E}Y^+/\mathbb{E}Y^- \geq z$ for some $z > 0$. Does it follow that $\mathbb{E}(X+Y)^+/\mathbb{E}(X+Y)^- \geq z$?

   Solution. It does not follow. Let $z = 1/3$. Define RV

$$X = \begin{cases} -1, & \text{wp } 1/2 \\ 1/3, & \text{wp } 1/2 \end{cases}$$

$$Y = \begin{cases} -1/3, & \text{wp } 1/2 \\ 1/9, & \text{wp } 1/2 \end{cases}$$

   where wp means 'with probability'. Hence,

$$X^+ = \begin{cases} 0, & \text{wp } 1/2 \\ 1/3, & \text{wp } 1/2 \end{cases}$$

$$Y^+ = \begin{cases} 0, & \text{wp } 1/2 \\ 1/9, & \text{wp } 1/2 \end{cases}$$

$$X^- = \begin{cases} 1, & \text{wp } 1/2 \\ 0, & \text{wp } 1/2 \end{cases}$$

$$Y^- = \begin{cases} 1/3, & \text{wp } 1/2 \\ 0, & \text{wp } 1/2 \end{cases}$$

   Hence $\mathbb{E}X^+/\mathbb{E}X^- = \frac{1/6}{1/2} = 1/3 \geq z$ and $\mathbb{E}Y^+/\mathbb{E}Y^- = \frac{1/18}{1/6} = 1/3 \geq z$. But,

$$X + Y = \begin{cases} -4/3 & \text{wp } 1/4 \\ -8/9 & \text{wp } 1/4 \\ 0 & \text{wp } 1/4 \\ 4/9 & \text{wp } 1/4 \end{cases}$$

$$(X+Y)^+ = \begin{cases} 0 & \text{wp } 3/4 \\ 4/9 & \text{wp } 1/4 \end{cases}$$

$$(X+Y)^- = \begin{cases} 4/3 & \text{wp } 1/4 \\ 8/9 & \text{wp } 1/4 \\ 0 & \text{wp } 1/2 \end{cases}$$

Hence, $\mathbb{E}(X+Y)^+/\mathbb{E}(X+Y)^- = \frac{1/9}{1/3+2/9} = 1/5 < z$. $\square$