1. Let $X \sim \text{Exp}(1)$ and $Y \sim N(0,1)$, independent of each other. Show that

$$X \overset{\mathcal{D}}{=} \sqrt{2X}|Y|$$

*Solution.* It suffices to show that $\mathbb{P}(\sqrt{2X}|Y| > t) = \mathbb{P}(X > t) = e^{-t}$. We have

$$\begin{aligned}
\mathbb{P}(\sqrt{2X}|Y| > t) &= \mathbb{P}\left(X > \frac{t^2}{2Y^2}\right) \\
&= \int_{-\infty}^{\infty} \mathbb{P}\left(X > \frac{t^2}{2y^2}\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} e^{-t^2/2y^2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t^2/y^2 + y^2)} dy \\
&= e^{-t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t/y - y)^2} dy \\
&= e^{-t} \int_{0}^{\infty} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(t/y - y)^2} dy
\end{aligned}$$

Call $\phi_t(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(t/y-y)^2}$. Notice that $\phi_t(y) = \phi_t(-\frac{t}{y})$. If we let $y = -\frac{t}{z}$ and $dy = \frac{t}{z^2}dz$, we have the identity

$$\int_0^\infty \phi_t(y) dy = \int_0^\infty \phi_t\left(-\frac{t}{z}\right) \frac{t}{z^2} dz = \int_0^\infty \phi_t(z) \frac{t}{z^2} dz = \int_0^\infty \phi_t(y) \frac{t}{y^2} dy$$

Now, let's prove that $\int_0^\infty \phi_t(y)dy$ is a constant function with respect to $t$. It suffices to show that

$$\frac{\partial}{\partial t} \int_0^\infty \phi_t(y) dy = 0$$

Indeed, since one may check $\frac{\partial}{\partial t}\phi_t(y) = \left(1 - \frac{t}{y^2}\right)\phi_t(y)$, and so we have

$$\begin{aligned}
\frac{\partial}{\partial t} \int_0^\infty \phi_t(y) dy &= \int_0^\infty \left(1 - \frac{t}{y^2}\right) \phi_t(y) dy \\
&= \int_0^\infty \phi_t(y) dy - \int_0^\infty \phi_t(y) \frac{t}{y^2} dy \\
&= 0
\end{aligned}$$

Hence, we may evaluate $\int_0^\infty \phi_t(y)dy$ by just setting $t = 0$. Notice that $\phi_0(y)$ is simply the density of the folded-normal distribution, and hence integrates to 1. Therefore,

$$\mathbb{P}(\sqrt{2X}|Y| > t) = e^{-t}$$

□

2. Recall that the probability generating function of $X$ taking values on $\{0, 1, \cdots\}$ is $g_X(t) = \mathbb{E}t^X = \sum_{x=0}^{\infty} p_X(x)t^x$. Using the table of distributions, how would one recover the PGF from the MGF? For $X$ being binomial, Poisson, and geometric, find $\mathbb{P}(X \text{ is even})$.

*Solution.* Notice that $M_X(t) = g_X(e^t)$, so simply replace $e^t$ in the expression for the MGF with just $t$ to get the PGF. For $X$ binomial, Poisson, and geometric, the PGF would be

$$(pt + q)^n, \qquad e^{\lambda(t-1)}, \qquad \frac{pt}{1 - qt}$$

respectively. Next, notice

$$g_X(1) = \sum_{x=0}^{\infty} p_X(x) = \sum_{x \in 2\mathbb{N}_0} p_X(x) + \sum_{x \in 2\mathbb{N}_0 + 1} p_X(x) = \mathbb{P}(X \text{ even}) + \mathbb{P}(X \text{ odd})$$
$$g_X(-1) = \sum_{x=0}^{\infty} p_X(x)(-1)^x = \sum_{x \in 2\mathbb{N}_0} p_X(x) - \sum_{x \in 2\mathbb{N}_0 + 1} p_X(x) = \mathbb{P}(X \text{ even}) - \mathbb{P}(X \text{ odd})$$

Adding both equations and dividing by 2, we get $\mathbb{P}(X \text{ even}) = \frac{g_X(-1) + g_X(1)}{2} = \frac{g_X(-1) + 1}{2}$. Hence, for $X$ binomial, Poisson, and geometric, the probabilities of being even is

$$\frac{1 + (1 - 2p)^n}{2}, \qquad \frac{1 + e^{-2\lambda}}{2}, \qquad \frac{1 - p}{2 - p}$$

**Follow-up:** Under what conditions will $\mathbb{P}(X \text{ even}) > \mathbb{P}(X \text{ odd})$? For binomial, this requires $(1 - 2p)^n > 0$, which is true whenever $p < \frac{1}{2}$ or $n$ even, or both. For Poisson, since $e^{-2\lambda} > 0$ always, Poisson will always give higher probability to being even. For geometric,

$$\frac{1 - p}{2 - p} < \frac{1 - p}{2 - 2p} = \frac{1}{2}$$

for all $p$, hence geometric (on $\{1, 2, \cdots\}$) will always give higher probability to being odd. $\square$

3. Suppose $X, Y$ are independent RV's taking on values in $\{1, \cdots, n\}$ with probabilities $p_X(\cdot)$ and $p_Y(\cdot)$, respectively. Is it possible to choose $p_X(1), \cdots, p_X(n), p_Y(1), \cdots, p_Y(n)$ such that $X + Y \sim \text{Unif}\{2, \cdots, 2n\}$?

*Solution.* Not possible. The statement would imply

$$\mathbb{P}(X + Y = k) = \frac{1}{2n-1}$$

for all $k \in \{2, \cdots, 2n\}$. Specifically,

$$\mathbb{P}(X + Y = 2) = p_X(1)p_Y(1) = \frac{1}{2n-1}, \qquad \mathbb{P}(X + Y = 2n) = p_X(n)p_Y(n) = \frac{1}{2n-1}$$

But then,

$$\begin{aligned}
\mathbb{P}(X + Y = n+1) &= \sum_{k=1}^{n} p_X(k)p_Y(n+1-k) \\
&= p_X(1)p_Y(n) + p_X(n)p_Y(1) + \sum_{k=2}^{n-1} p_X(k)p_Y(n+1-k) \\
&\geq p_X(1)p_Y(n) + p_X(n)p_Y(1) \\
&\geq 2\sqrt{p_X(1)p_Y(n)p_X(n)p_Y(1)} && \text{AM-GM inequality} \\
&= \frac{2}{2n-1} \\
&> \frac{1}{2n-1}
\end{aligned}$$

Recall the AM-GM inequality states, for $a, b > 0$, that $\frac{a+b}{2} \geq \sqrt{ab}$. Above, we let $a = p_X(1)p_Y(n)$ and $b = p_X(n)p_Y(1)$. Hence, the steps above show that $\mathbb{P}(X + Y = n+1)$ will always be greater than (and hence can never equal) $1/(2n-1)$ (unless trivially $n = 1$). $\square$

4. Suppose $X$ and $Y$ are RV's satisfying

- $X, Y \geq 0$
- $X$ and $Y$ are independent and identically distributed, with $\mathbb{P}(X = 0) < 1$
- $\min(X, Y) \stackrel{\mathcal{D}}{=} \frac{X}{2}$

Prove that $X$ and $Y$ must be exponentially distributed.

*Solution.* Let $S(t) = \mathbb{P}(X > t)$. Then,

$$\begin{aligned} S^2(t) &= (\mathbb{P}(X > t))^2 \\ &= \mathbb{P}(X > t) \cdot \mathbb{P}(X > t) \\ &= \mathbb{P}(X > t) \cdot \mathbb{P}(Y > t) && \text{by } X, Y \text{ iid} \\ &= \mathbb{P}(X > t, Y > t) && \text{by } X, Y \text{ iid} \\ &= \mathbb{P}(\min(X, Y) > t) \\ &= \mathbb{P}\left(\frac{X}{2} > t\right) && \text{by } \min(X, Y) \stackrel{\mathcal{D}}{=} \frac{X}{2} \\ &= S(2t) \end{aligned}$$

Now let's induct $S^{2n}(t) = S(2nt)$ for all positive integers $n$ and nonnegative $t$. We just showed the base case $n = 1$ above. For inductive step, we have

$$S^{2(n+1)}(t) = S^{2n}(t)S^2(t) = S(2nt)S(2t)$$

divide both sides by $S(2t)$ to get $S^{2n}(t) = S(2nt)$. Next, we see that

$$S^{2n}\left(\frac{m}{n}t\right) = S\left(2n \cdot \frac{m}{n}t\right) = S(2mt) = S^{2m}(t)$$

hence $S^{m/n}(t) = S(\frac{m}{n}t)$ for any positive integers $m, n$. This means that $S^r(t) = S(rt)$ for any positive rational $r$. Next, let's prove that $S(t)$ is continuous. For fixed $t$, let

$$\delta = -\frac{\log(1 + \frac{\epsilon}{S(t)})}{\log(S(t))}$$

and suppose $t'$ satisfies $|t - t'| < \delta$. In the case $t' < t < t' + \delta$, we have

$$t - t' < -\frac{\log(1 + \frac{\epsilon}{S(t)})}{\log(S(t))} \implies S^{t'}(t) < (S(t) + \epsilon)^t$$

from which we can manipulate into

$$\frac{\log(S(t)+\epsilon)}{\log(S(t))} < \frac{t'}{t}$$

By denseness of the rationals over the real numbers, we can find an $r$ such that

$$\frac{\log(S(t)+\epsilon)}{\log(S(t))} < r < \frac{t'}{t}$$

The lower bound can be manipulated into $S^r(t) - S(t) < \epsilon$. Since $S(t)$ is non-increasing, $S(t') - S(t) \geq 0$ and

$$S(t') - S(t) \leq S(rt) - S\left(r\right) = S^r(t) - S(t) < \epsilon$$

The proof is analogous for $t < t' < t + \epsilon$. Hence, $S(t)$ is continuous. Since

$$S(t) = S^t(1) = e^{t \log S(1)}$$

for any real $t$, we see that $S(\cdot)$ is the survival function of an exponential with $\lambda = -\log S(1)$.

$\square$

5. Consider $n \times n$ matrix $M$, each of its entries randomly selected from $\{-1, 1\}$ with probability $1/2$. Show that $\mathbb{E}\det(M) = 0$ and $\mathrm{var}(\det(M)) = n!$. Use this result to show that, for $n = 7$, there exists some configuration of entries from $\{-1, 1\}$ into $M$ such that $\det(M)$ is at least 71.

*Solution.* Let the entries of $M$ be $m_{ij}$. Recall one definition of determinant is

$$\det(M) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$$

where $S_n$ is the set of all bijective functions from $\{1, \cdots, n\}$ to itself, known as the *symmetric group*. Hence, for a function $\sigma \in S_n$, the vector $(\sigma(1), \cdots, \sigma(n))$ is just a different permutation of $(1, \cdots, n)$. Hence,

$$\begin{aligned}
\mathbb{E}\det(M) &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \mathbb{E} m_{1,\sigma(1)} \cdots m_{n,\sigma(n)} \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \mathbb{E} m_{1,\sigma(1)} \cdots \mathbb{E} m_{n,\sigma(n)} \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot 0 \\
&= 0
\end{aligned}$$

since $\mathbb{E} m_{ij} = 0$ for any $i, j$. Next,

$$\begin{aligned}
\mathrm{var}(\det(M)) &= \mathbb{E}(\det(m))^2 - (\mathbb{E}\det(M))^2 \\
&= \mathbb{E}\left(\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}\right)\left(\sum_{\tau \in S_n} \mathrm{sgn}(\tau) m_{1,\tau(1)} \cdots m_{n,\tau(n)}\right) \\
&= \sum_{\sigma,\tau \in S_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \mathbb{E} m_{1,\sigma(1)} \cdots m_{n,\sigma(n)} m_{1,\tau(1)} \cdots m_{n,\tau(n)}
\end{aligned}$$

Let $\mathcal{J} = \{i : \sigma(i) = \tau(i)\}$. Then for $i \in \mathcal{J}$, we have $m_{i,\sigma(i)} m_{i,\tau(i)} = 1$, and for all $i \notin \mathcal{J}$, it remains the product of two iid RV's. Hence,

$$m_{1,\sigma(1)} \cdots m_{n,\sigma(n)} m_{1,\tau(1)} \cdots m_{n,\tau(n)} = \prod_{j \notin \mathcal{J}} m_{i,\sigma(i)} m_{i,\tau(i)}$$

which will have expectation 0, unless $\mathcal{J} = \{1, \cdots, n\}$, Hence, the double sum $\sum_{\sigma,\tau \in S_n}$ is mostly summing a bunch of zeros except when $\sigma = \tau$, in which case it's summing ones.

Thus,

$$\operatorname{var}(\det(M)) = \sum_{\sigma \in S_n} 1 = |S_n| = n!$$

Now for the second part of the question, note that finding such a configuration means that $\max_{m_{ij}\in\{0,1\}} \det(M) \geq 71$. Without loss of generality, we may assume $\det(M) > 0$, since we can negate a row in $M$, which switches the sign of the determinant. Since $\det(M) \mapsto (\det(M))^2$ is monotone increasing for positive $\det(M)$, maximizing $\det(M)$ is the same objective as maximizing $(\det(M))^2$. Hence,

$$\begin{aligned}\max_{m_{ij}\in\{0,1\}} \det(M) &= \sqrt{\max_{m_{ij}\in\{0,1\}} (\det(M))^2} \\ &\geq \sqrt{\mathbb{E}(\det(M))^2} \\ &= \sqrt{n!}\end{aligned}$$

For $n = 7$, $\max_{m_{ij}\in\{0,1\}} \det(M) \geq \sqrt{7!} \doteq 70.99$. Since $\det(M)$ must be an integer, we have that $\max_{m_{ij}\in\{0,1\}} \det(M) \geq 71$. □