Lab 4

Problem Solving with Probability Distributions

4.1. Discovering PMF's/PDF's

For this lab, we will denote Distribution(x; Parameters) as the density/mass function associated with the distribution name with its respective parameters. For example, $N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$. Refer to the table of distributions at the end of this lab for the abbreviation of distribution names. If we are not referring to any specification distribution, we will use our $f_X(x)$ notation from before.

For any positive function $g(x)$ that has finite integral $\int_{\mathbb{R}} g(x)dx = C$, a valid PDF can be designed by taking $f_X(x) = g(x)/C$, and hence C has the interpretation of a normalizing constant. In some sense, this gives you a “recipe” in designing whatever PMF/PDF you want, and personally I feel many historic PMF/PDF's were designed by taking named identities and dividing by the normalizing constant. For example, the *Chu-Vandermonde identity* states

$$\sum_{x=0}^{K} \binom{M}{x} \binom{N-M}{K-x} = \binom{N}{K}$$

If we divide both sides by $\binom{N}{K}$, the summands then become the PMF for the hypergeometric distribution! You can observe this for all the other PMF/PDF's; verifying the integrate or sum to 1 becomes a game in actually proving some historical identity equation.

4.2. Strategies

Case study 1: Computing Moments. There are generally three strategies for this. One strategy, if the MGF is simple enough, is to use that. Otherwise...

Example 4.2.1. (Manipulating into a PMF/PDF) Find the moments of the gamma and beta distributions.

Solution. Let $X \sim \text{Gamma}(\alpha, \beta)$ and $Y \sim \text{Beta}(\alpha, \beta)$. Realize that

$$\begin{aligned} x^k \cdot \text{Gamma}(x; \alpha, \beta) &= x^k \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \\ &= \frac{\Gamma(k+\alpha)\beta^{k+\alpha}}{\Gamma(\alpha)\beta^\alpha} \left(\frac{1}{\Gamma(k+\alpha)\beta^{k+\alpha}} x^{k+\alpha-1} e^{-\frac{x}{\beta}} \right) \\ &= \frac{\Gamma(k+\alpha)\beta^k}{\Gamma(\alpha)} \cdot \text{Gamma}(x; k+\alpha, \beta) \end{aligned}$$

Hence, when we integrate both sides, the LHS becomes $\mathbb{E}X^k$, while the $\text{Gamma}(x; k+\alpha, \beta)$ on the RHS integrates to 1 and we are left with $\frac{\Gamma(k+\alpha)\beta^k}{\Gamma(\alpha)}$. Also,

$$\begin{aligned} y^k \cdot \text{Beta}(y; \alpha, \beta) &= y^k \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} \\ &= \frac{B(k+\alpha, \beta)}{B(\alpha, \beta)} \left(\frac{1}{B(k+\alpha, \beta)} y^{k+\alpha-1}(1-y)^{\beta-1} \right) \\ &= \frac{B(k+\alpha, \beta)}{B(\alpha, \beta)} \cdot \text{Beta}(y; k+\alpha, \beta) \end{aligned}$$

Again, integrating both sides gives $\mathbb{E}Y^k$ on the LHS and $\frac{B(k+\alpha,\beta)}{B(\alpha,\beta)}$ on the RHS. □

Example 4.2.2. (Integration by parts) Let $X \sim \text{Exp}(\lambda)$ and $Y \sim N(0, 1)$. Compute $\mathbb{E}X^k$ and $\mathbb{E}Y^k$ for positive integers k.

Solution. We have

$$\begin{aligned} \mathbb{E}X^k &= \int_0^\infty x^k \lambda e^{-\lambda x} dx \\ &= \left\{ -x^k e^{-\lambda x} \right\}_0^\infty - \int_0^\infty -k x^{k-1} e^{-\lambda x} dx \\ &= \frac{k}{\lambda} \int_0^\infty \lambda x^{k-1} e^{-\lambda x} dx \\ &= \frac{k}{\lambda} \mathbb{E}X^{k-1} \end{aligned}$$

We can perform the expression recursively to get $\mathbb{E}X^k = \frac{k!}{\lambda^k}$. (Compare this with Gamma example.)

For the normal distribution, notice that $y^k \cdot \text{N}(y; 0, 1)$, when k is odd, is an odd function, and hence the integral over the real line gives us 0. Hence, $\mathbb{E}Y^k = 0$ for odd k. For even k, let $k = 2m$. We first note the fact that

$$\begin{aligned} \frac{d}{dy} \text{N}(y; 0, 1) &= \frac{d}{dy} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \\ &= -y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \\ &= -y \cdot \text{N}(y; 0, 1) \end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}Y^{2m} &= \int_{-\infty}^{\infty} y^{2m} \cdot \mathrm{N}(y;0,1)dy \\
&= \int_{-\infty}^{\infty} -y^{2m-1} \cdot \left(\frac{d}{dy}\mathrm{N}(y;0,1)\right)dy \\
&= \int_{-\infty}^{\infty} -y^{2m-1} \cdot d\mathrm{N}(y;0,1) \\
&= \left\{-y^{2m-1}\mathrm{N}(y;0,1)\right\}_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2m-1)y^{2m-2} \cdot \mathrm{N}(y;0,1)dy \\
&= (2m-1)\mathbb{E}Y^{2m-2}
\end{aligned}$$

Again, recursively apply the relation to get

$$\mathbb{E}Y^{2m} = (2m-1)\cdot(2m-3)\cdots 3\cdot 1 = (2m-1)!!$$

Hence, for k even, we have $\mathbb{E}Y^k = (k-1)!!$. □

Case study 2: Exponential distribution.

Example 4.2.3. Let $X_i \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda_i)$ for $i = 1, \cdots, n$.

(a) Show that $\xi_n \overset{\text{def}}{=} \min(X_1, \cdots, X_n) \sim \mathrm{Exp}(\sum_{i=1}^n \lambda_i)$.

(b) Find the density of $\eta = X_1 - X_2$.

(c) Show that ξ_2 and η are independent.

Solution. We have

(a)

$$\mathbb{P}(\xi_n > x) = \mathbb{P}(X_1 > x, \cdots, X_n > x) = \prod_{i=1}^n \mathbb{P}(X_i > x) = \prod_{i=1}^n e^{-\lambda_i x} = e^{-(\sum_{i=1}^n \lambda_i)x}$$

(b) The most aesthetic solution perfectly exploits the memoryless property. First suppose $x < 0$. Then,

$$\begin{aligned}
F_\eta(x) &= \mathbb{P}(X_1 - X_2 \le x) \\
&= \mathbb{P}(X_2 \ge X_1 - x) \\
&= \mathbb{P}(X_2 \ge X_1 - x | X_2 \ge X_1)P(X_2 \ge X_1) \\
&= \mathbb{P}(X_2 \ge -x)\frac{\lambda_1}{\lambda_1 + \lambda_2} \\
&= e^{\lambda_2 x}\frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

Similarly, for $x \geq 0$,

$$\begin{aligned}F_\eta(x) &= \mathbb{P}(X_1 - X_2 \leq x) \\ &= 1 - \mathbb{P}(X_1 > X_2 + x) \\ &= 1 - \mathbb{P}(X_1 > X_2 + x | X_1 \geq X_2) P(X_1 \geq X_2) \\ &= 1 - \mathbb{P}(X_1 > x)\frac{\lambda_2}{\lambda_1 + \lambda_2} \\ &= 1 - e^{-\lambda_1 x}\frac{\lambda_2}{\lambda_1 + \lambda_2}\end{aligned}$$

Hence

$$f_\eta(x) = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} \cdot \begin{cases} e^{\lambda_2 x}, & x \leq 0 \\ e^{-\lambda_1 x}, & x \geq 0 \end{cases}$$

(c) It suffices to show that $\mathbb{P}(\xi_2 > a, \eta > b) = \mathbb{P}(\xi_2 > a)\mathbb{P}(\eta > b)$. First suppose $b \geq 0$. We have

$$\begin{aligned}\mathbb{P}(X_1 > a, X_2 > a, X_1 > X_2 + b) &= \mathbb{P}(X_2 > a, X_1 > X_2 + b) \\ &= \int_0^\infty \mathbb{P}(x_2 > a, X_1 > x_2 + b) f_{X_2}(x_2) dx_2 \\ &= \int_a^\infty \mathbb{P}(X_1 > x_2 + b) f_{X_2}(x_2) dx_2 \\ &= \int_a^\infty e^{-\lambda_1(x_2 + b)} \lambda_2 e^{-\lambda_2 x_2} dx_2 \\ &= \left(e^{-(\lambda_1+\lambda_2)a}\right)\left(\frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 b}\right) \\ &= \mathbb{P}(\xi_2 > a)\mathbb{P}(\eta > b)\end{aligned}$$

And for $b \leq 0$,

$$\begin{aligned}\mathbb{P}(X_1 > a, X_2 > a, X_1 > X_2 + b) &= \mathbb{P}(X_1 > a, a < X_2 < X_1 - b) \\ &= \int_0^\infty \mathbb{P}(x_1 > a, a < X_2 < x_1 - b) f_{X_1}(x_1) dx_1 \\ &= \int_a^\infty \mathbb{P}(a < X_2 < x_1 - b) f_{X_1}(x_1) dx_1 \\ &= \int_a^\infty \left(e^{-\lambda_2 a} - e^{-\lambda_2(x_1 - b)}\right) \lambda_1 e^{-\lambda_1 x_1} dx_1 \\ &= e^{-(\lambda_1+\lambda_2)a}\left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{\lambda_2 b}\right) \\ &= \mathbb{P}(\xi_2 > a)\mathbb{P}(\eta > b)\end{aligned}$$

□

Distributions

$Name$	S	$f_X(x)$	$E_X(X)$	$\mathrm{Var}_X(X)$	$\mathcal{M}_X(t)$
$\mathrm{Unif}\{a,b\}$	$\{a,...,b\}$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$	$\frac{e^{(b+1)t}-e^{at}}{n(e^t-1)}$
$\mathrm{Ber}(p)$	$\{0,1\}$	$p^x q^{1-x}$	p	pq	pe^t+q
$\mathrm{Bin}(n,p)$	$\{0,...,n\}$	$\binom{n}{x} p^x q^{n-x}$	np	npq	$(pe^t+q)^n$
$\mathrm{Geo}(p)$	$\{1,...\}$	pq^{x-1}	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
$\mathrm{NBin}(r,p)$	$\{s,s+1,...\}$	$\binom{x-1}{s-1} p^s q^{x-s}$	$\frac{s}{p}$	$\frac{sq}{p^2}$	$(\frac{pe^t}{1-qe^t})^r$
$\mathrm{HGeo}(M,N,K)$	$K-(N-M) \leq x \leq M$	$\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$	$\frac{KM}{N}$	$\frac{K(N-K)\ M(N-M)}{N^2(N-1)}$	detailed
$\mathrm{Pois}(\lambda)$	$\{0,1,...\}$	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$
$\mathrm{Unif}(a,b)$	$[a,b]$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{t(b-a)}$
$\mathrm{Beta}(\alpha,\beta)$	$[0,1]$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	detailed
$\mathrm{Gamma}(\alpha,\beta)$	$\mathbb{R}^+$	$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$	$\alpha\beta$	$\alpha\beta^2$	$(\frac{1}{1-\beta t})^\alpha$
$\chi^2(p)$	$\mathbb{R}^+$	$\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{p/2}} x^{p/2-1} e^{-x/2}$	p	$2p$	$\left(\frac{1}{1-2t}\right)^{\frac{p}{2}}$
$\mathrm{Exp}(\beta)$	$\mathbb{R}^+$	$\frac{1}{\beta} e^{-y/\beta}$	β	β^2	$\frac{1}{1-\beta t}$
$\mathrm{Weibull}(\gamma,\beta)$	$\mathbb{R}^+$	$\frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta}$	$\beta^{1/\gamma}\Gamma\left(1+\frac{1}{\gamma}\right)$	$\beta^{2/\gamma}\left[\Gamma\left(1+\frac{2}{\gamma}\right)-\Gamma^2\left(1+\frac{1}{\gamma}\right)\right]$	detailed
$\mathrm{Pareto}(\alpha,\beta)$	$\mathbb{R} > \alpha$	$\frac{\beta\alpha^\beta}{x^{\beta+1}}$	$\frac{\beta\alpha}{\beta-1}$	$\frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}$	$\times$
$\mathrm{N}(\mu,\sigma^2)$	$\mathbb{R}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$	μ	σ^2	$e^{\mu t+\frac{t^2 a^2}{2}}$
$\mathrm{LogNorm}(\mu,\sigma^2)$	$\mathbb{R}$	$\frac{1}{x\sigma\sqrt{(2\pi)}} e^{-\frac{\left(\frac{\log(x)-\mu}{\sigma}\right)^2}{2}}$	$e^{\mu+\sigma^2 2}$	$e^{2(\mu+\sigma^2)}-e^{2\mu+\sigma^2}$	$E[X^n]=e^{n\mu+n^2\sigma^2/2}$
$t(\nu)$	$\mathbb{R}$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0	$\frac{\nu}{\nu-2}$	$\times$
$\mathrm{Cauchy}(\theta,a)$	$\mathbb{R}$	$\frac{1}{a\pi\left(1-\left(\frac{x-\theta}{a}\right)^2\right)}$	$\times$	$\times$	$\times$

4.3. Additional Problems

1. Let $X \sim \mathrm{Exp}(1)$ and $Y \sim N(0,1)$, independent of each other. Show that

$$X \stackrel{\mathcal{D}}{=} \sqrt{2X}|Y|$$

2. Recall that the probability generating function of X taking values on $\{0, 1, \cdots\}$ is $g_X(t) = \mathbb{E}t^X = \sum_{x=0}^{\infty} p_X(x)t^x$. Using the table of distributions, how would one recover the PGF from the MGF? For X being binomial, Poisson, and geometric, find $\mathbb{P}(X \text{ is even})$.

3. Suppose X, Y are independent RV's taking on values in $\{1, \cdots, n\}$ with probabilities $p_X(\cdot)$ and $p_Y(\cdot)$, respectively. Is it possible to choose $p_X(1), \cdots, p_X(n), p_Y(1), \cdots, p_Y(n)$ such that $X + Y \sim \text{Unif}\{2, \cdots, 2n\}$?

4. Suppose X and Y are RV's satisfying

   - $X, Y \geq 0$
   - X and Y are independent and identically distributed, with $\mathbb{P}(X = 0) < 1$
   - $\min(X, Y) \stackrel{\mathcal{D}}{=} \frac{X}{2}$

   Prove that X and Y must be exponentially distributed.

5. Consider $n \times n$ matrix M, each of its entries randomly selected from $\{-1, 1\}$ with probability 1/2. Show that $\mathbb{E} \det(M) = 0$ and $\mathrm{Var}(\det(M)) = n!$. Use this result to show that, for $n = 7$, there exists some configuration of entries from $\{-1, 1\}$ into M such that $\det(M)$ is at least 71.