1. To get in the "mood" for inequalities to come, prove the following elementary inequalities:

   - $-\log(1-u) - u \le \frac{u^2}{2(1-u)}$ for $u \in (0,1)$
   - $\log(1-\theta+\theta e^u) - \theta u \le \frac{1}{8}u^2$ for $u, \theta > 0$
   - $$\min_{1\le i\le n} \frac{a_i}{b_i} \le \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \le \max_{1\le i\le n} \frac{a_i}{b_i}$$

     for any positive $a_1, \cdots, a_n, b_1, \cdots, b_n$.

   Solution. For each one,

   - Using the fact that $\frac{u^k}{k} \le \frac{u^k}{2}$ for all $k \ge 2$ and $u \in (0,1)$, we employ power series expansion here:

     $$-\log(1-u) = \sum_{k=1}^{\infty} \frac{u^k}{k} = u + \sum_{k=2}^{\infty} \frac{u^k}{k} \le u + \sum_{k=2}^{\infty} \frac{u^k}{2} = u + \frac{u^2}{2(1-u)}$$

   - Define $h(u) = \log(1-\theta+\theta e^u) - \theta u$. Taylor's theorem states that

     $$h(u) = h(0) + h'(0)u + \frac{h''(\xi)}{2}u^2$$

     for some $\xi \in (0,u)$. We calculate

     $$\begin{aligned} h(0) &= 0 \\ h'(u) = \frac{\theta e^u}{1-\theta+\theta e^u} - \theta \implies h'(0) &= 0 \\ h''(u) = \frac{\theta e^u(1-\theta)}{(1-\theta+\theta e^u)^2} = p(1-p) \implies h''(\xi) &\le \frac{1}{4} \end{aligned}$$

     where $p = \theta e^u/(1-\theta+\theta e^u)$. The representation $p(1-p)$ makes it clear that h'' is bounded above by 1/4, so certainly $h''(\xi) \le \frac{1}{4}$. Therefore,

     $$h(u) \le 0 + 0 \cdot u + \frac{1}{2\cdot 4}u^2 = \frac{1}{8}u^2$$

   - Denote $M = \max_{1\le i\le n} \frac{a_i}{b_i}$ and $m = \min_{1\le i\le n} \frac{a_i}{b_i}$. For the upper inequality,

     $$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \cdot \frac{a_i}{b_i} \le \sum_{i=1}^n b_i \cdot M$$

and then divide both sides by $\sum_{i=1}^{n} b_i$. The proof is completely analogous for the lower inequality.

$\square$

2. Let X have finite second moment. Denote $\mu = \mathbb{E}X, \sigma^2 = \text{var}(X)$, and $\widetilde{x}$ as the median of X. Prove that

$$|\mu - \widetilde{x}| \leq \sigma$$

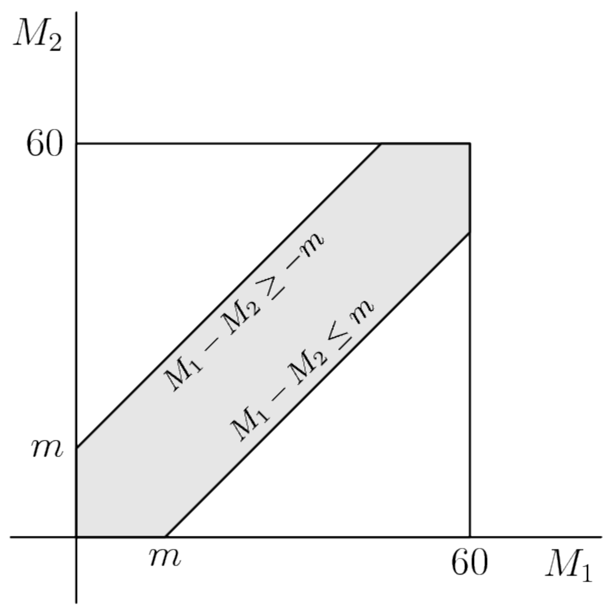
Solution. We have

$$\begin{aligned}
|\mu - \widetilde{x}| &= |\mathbb{E}(X - \widetilde{x})| \\
&\leq \mathbb{E}|X - \widetilde{x}| && \text{Triangle inequality} \\
&\leq \mathbb{E}|X - \mu| && \text{Since } \min_b \mathbb{E}|X - b| = \mathbb{E}|X - \widetilde{x}| \\
&= \sqrt{\mathbb{E}(X-\mu)^2 - \text{Var}(|X - \mu|)} \\
&\leq \sqrt{\mathbb{E}(X-\mu)^2} \\
&= \sigma
\end{aligned}$$

We could've also immediately concluded that $\mathbb{E}|X - \mu| \leq \sqrt{\mathbb{E}(X-\mu)^2}$ by Jensen's inequality, but we haven't gone over this in class yet. □

3. (The "random meeting" problem.) Person A and person B have agreed to meet between 7:00 p.m. and 8:00 p.m. at a particular location. They have both forgotten the exact meeting time and choose their respective arrival times randomly and independently from each other between 7:00 p.m. and 8:00 p.m., according to the uniform distribution on the interval [7:00, 8:00]. They both have patience to wait no longer than 10 min. Prove that the probability that A and B will actually meet equals 11/36.

   Solution. Without loss of generality, let's assume person A and person B randomly arrive on the interval $[0, 60]$. Let $M_1, M_2 \overset{\text{iid}}{\sim} \text{Unif}(0, 60)$ represent the minute at which each arrive. In order for them to meet, we must have $|M_1 - M_2| \leq m$, where $m = 10$. Because of this nice uniform distribution, calculating $\mathbb{P}(|M_1 - M_2| \leq m)$ is equivalent to calculating the area of the shaded region, divided by area of the 60×60 region.



   More simply, we can calculate $1 - \mathbb{P}(|M_1 - M_2| > m)$, where $\mathbb{P}(|M_1 - M_2| > m)$ corresponds to the two triangles outside the shaded region, which combined, become a $(60-m)\times(60-m)$ square. So we have

$$\mathbb{P}(|M_1 - M_2| \leq m) = 1 - \frac{(60 - m)^2}{60^2} = 1 - \frac{50^2}{60^2} = \frac{11}{36}$$

□

4. Suppose $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$. Show that $X|(X+Y=n) \sim \text{Bin}(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$.

Solution. We note that $X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$. Hence

$$\begin{aligned}
\mathbb{P}(X = k | X + Y = n) &= \frac{\mathbb{P}(X = k, X + Y = n)}{\mathbb{P}(X + Y = n)} \\
&= \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(X + Y = n)} \\
&= \frac{\frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!}}{\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}} \\
&= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}
\end{aligned}$$

which is the PMF of $\text{Bin}(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$. □

5. Suppose X, Y satisfy $\mathbb{E}|X|^2, \mathbb{E}|Y|^2 < \infty$ and that $\mathbb{E}(X|Y) = Y, \mathbb{E}(Y|X) = X$. Prove that $X \overset{\text{a.s}}{=} Y$.

Remark: The statement is still true when $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$, although the problem is much harder.

Solution. We will use of the law of total expectation: $\mathbb{E}(\mathbb{E}(A|B)) = \mathbb{E}A$. First,

$$\mathbb{E}XY = \mathbb{E}(\mathbb{E}(XY|X)) = \mathbb{E}(X\mathbb{E}(Y|X)) = \mathbb{E}X^2$$

hence

$$\mathbb{E}(X-Y)^2 = \mathbb{E}X^2 - 2\mathbb{E}XY + \mathbb{E}Y^2 = \mathbb{E}Y^2 - \mathbb{E}X^2$$

Similarly, $\mathbb{E}XY = \mathbb{E}Y^2$, and we get $\mathbb{E}(X-Y)^2 = \mathbb{E}X^2 - \mathbb{E}Y^2$. This implies that

$$\mathbb{E}(X-Y)^2 = -\mathbb{E}(X-Y)^2 \implies \mathbb{E}(X-Y)^2 = 0$$

Next, by Markov's inequality,

$$\begin{aligned} \mathbb{P}((X-Y)^2 \geq \epsilon) &\leq \frac{\mathbb{E}(X-Y)^2}{\epsilon} \\ &= 0 \end{aligned}$$

which implies that $\mathbb{P}((X-Y)^2 < \epsilon) = 1$. Since $\epsilon > 0$ is arbitrary, we have that $\mathbb{P}((X-Y)^2 = 0) = 1$, hence $P(X = Y) = 1$, which is the definition of $X \overset{\text{a.s}}{=} Y$. $\square$