1. Suppose X satisfies $\mathbb{E}X^2 < \infty$. Find the maximum value of $\mathbb{E}ZX$ subject to $Z \geq 0$ and $\mathbb{E}Z^2 \leq 1$.

   Solution. We have

$$\begin{aligned}
\mathbb{E}ZX &= \mathbb{E}ZX\mathbb{I}(X < 0) + \mathbb{E}ZX\mathbb{I}(X \geq 0) \\
&\leq \mathbb{E}ZX\mathbb{I}(X \geq 0) && \text{Since } Z \geq 0 \\
&\leq (\mathbb{E}Z^2)^{\frac{1}{2}}(\mathbb{E}X^2\mathbb{I}(X \geq 0))^{\frac{1}{2}} && \text{Cauchy-Schwarz} \\
&\leq (\mathbb{E}X^2\mathbb{I}(X \geq 0))^{\frac{1}{2}} && \text{Since } \mathbb{E}Z^2 \leq 1
\end{aligned}$$

   Hence, we have an upper bound of $(\mathbb{E}X^2\mathbb{I}(X \geq 0))^{\frac{1}{2}}$. To attain equality in the bound, we must find Z such that all the inequalities become equalities. In order to attain these equalities, Z must satisfy

   - $Z = 0$ whenever $X < 0$.
   - Z is proportional to $X\mathbb{I}(X \geq 0)$
   - $\mathbb{E}Z^2 = 1$

   We may check that $Z = X\mathbb{I}(X \geq 0)/(\mathbb{E}X^2\mathbb{I}(X \geq 0))^{\frac{1}{2}}$ indeed satisfies these properties, and upon plugging into the expression, we do indeed attain this bound. Hence, $\mathbb{E}ZX$ attains its max at this value. □

2. Let X be a nonnegative random variable with finite and strictly positive second moment. Prove the *Paley–Zygmund inequality*: for all $\varepsilon \in [0, 1]$,

$$\mathbb{P}(X > \varepsilon \mathbb{E}X) \geq (1-\varepsilon)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}$$

Solution. We have

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}X\mathbb{I}(X \leq \varepsilon \mathbb{E}X) + \mathbb{E}X\mathbb{I}(X > \varepsilon \mathbb{E}X) \\
&\leq \varepsilon \mathbb{E}X + \mathbb{E}X\mathbb{I}(X > \varepsilon \mathbb{E}X) \\
&\leq \varepsilon \mathbb{E}X + (\mathbb{E}X^2)^{\frac{1}{2}} (\mathbb{E}\mathbb{I}^2(X > \varepsilon \mathbb{E}X))^{\frac{1}{2}} && \text{Cauchy-Schwarz} \\
&= \varepsilon \mathbb{E}X + (\mathbb{E}X^2)^{\frac{1}{2}} (\mathbb{P}(X > \varepsilon \mathbb{E}X))^{\frac{1}{2}}
\end{aligned}$$

Solving for $\mathbb{P}(X > \varepsilon \mathbb{E}X)$ yields the desired inequality. □

3. This is a variation of problem B4 from the 2013 Putnam Mathematical Competition. Let X and Y be bounded RV's, say $|X| \leq M_X$ and $|Y| \leq M_Y$. They are not necessarily independent. Prove that

$$\text{Var}(XY) \leq 2M_X^2 \text{Var}(Y) + 2M_Y^2 \text{Var}(X)$$

Perhaps the best way to prove this is through the following steps:

$$\begin{aligned}
\text{Var}(XY) &\leq \mathbb{E}(XY - \mathbb{E}X\mathbb{E}Y)^2 && \text{(a)} \\
&\leq 2\mathbb{E}(XY - X\mathbb{E}Y)^2 + 2\mathbb{E}(X\mathbb{E}Y - \mathbb{E}X\mathbb{E}Y)^2 && \text{(b)} \\
&\leq 2M_X^2 \text{Var}(Y) + 2M_Y^2 \text{Var}(X) && \text{(c)}
\end{aligned}$$

Prove steps (a) – (c).

Solution. We have

$$\begin{aligned}
\text{Var}(XY) &= \mathbb{E}(XY - \mathbb{E}XY)^2 \\
&\leq \mathbb{E}(XY - \mathbb{E}X\mathbb{E}Y)^2 && \text{since } \min_a \mathbb{E}(Z-a)^2 = \mathbb{E}(Z - \mathbb{E}Z)^2 \\
&= \mathbb{E}(XY - X\mathbb{E}Y + X\mathbb{E}Y - \mathbb{E}X\mathbb{E}Y)^2 \\
&\leq 2\mathbb{E}(XY - X\mathbb{E}Y)^2 + 2\mathbb{E}(X\mathbb{E}Y - \mathbb{E}X\mathbb{E}Y)^2 && \text{since } \mathbb{E}(A+B)^2 \leq 2\mathbb{E}A^2 + 2\mathbb{E}B^2 \\
&\leq 2M_X^2\mathbb{E}(Y - \mathbb{E}Y)^2 + 2M_Y^2\mathbb{E}(X - \mathbb{E}X)^2 \\
&= 2M_X^2 \text{Var}(Y) + 2M_Y^2 \text{Var}(X)
\end{aligned}$$

□

4. Let X, U, V be RV's with finite second moments. We know that $\rho_{UX} \leq 1$ and $\rho_{VX} \leq 1$, and hence we can instantly conclude $\rho_{UX} + \rho_{VX} \leq 2$ and $\rho_{UX}\rho_{VX} \leq 1$. These bound are actually quite weak; if U and V are not completely correlated, then it is impossible for both ρ_{UX} and ρ_{VX} to simultaneously equal 1. We shall derive improved bounds in this problem.

   (a) Let $\widetilde{X} = (X - \mu_x)/\sigma_X$. Show that $\mathbb{E}(\widetilde{X} - \widetilde{U})^2 = 2 - 2\rho_{UX}$.[1]

   (b) Prove the identity

$$2\mathbb{E}A^2 + 2\mathbb{E}B^2 = \mathbb{E}(A+B)^2 + \mathbb{E}(A-B)^2$$

   (c) Using (a) and (b), prove that

$$\rho_{UX} + \rho_{VX} \leq \frac{3 + \rho_{UV}}{2}$$

   Hint: Let $A = \widetilde{X} - (\widetilde{U} + \widetilde{V})/2$ and $B = (\widetilde{U} - \widetilde{V})/2$.

   (d) Define $W = 2\frac{\sigma_{UX}}{\sigma_X^2}X - U$. Show that $\sigma_W^2 = \sigma_U^2$; hence $\sigma_{WV} \leq \sigma_U \sigma_V$. Reorganize the latter to show that

$$\rho_{UX}\rho_{VX} \leq \frac{1 + \rho_{UV}}{2}$$

Solution.

   (a) $\mathbb{E}(\widetilde{X} - \widetilde{U})^2 = \mathbb{E}\widetilde{X}^2 + \mathbb{E}\widetilde{U}^2 - 2\mathbb{E}\widetilde{X}\widetilde{U} = 1 + 1 - 2\frac{\mathbb{E}(X-\mu_X)(U-\mu_U)}{\sigma_X\sigma_U} = 2 - 2\rho_{UX}$

   (b)

$$\begin{aligned}\mathbb{E}(A+B)^2 + \mathbb{E}(A-B)^2 &= (\mathbb{E}A^2 + \mathbb{E}B^2 + 2\mathbb{E}AB) + (\mathbb{E}A^2 + \mathbb{E}B^2 - 2\mathbb{E}AB) \\ &= 2\mathbb{E}A^2 + 2\mathbb{E}B^2\end{aligned}$$

   (c) From part (b), and since $\mathbb{E}A^2 \geq 0$, we have

$$2\mathbb{E}B^2 \leq \mathbb{E}(A+B)^2 + \mathbb{E}(A-B)^2$$

   Applying the hint,

$$2\mathbb{E}\left(\frac{\widetilde{U} - \widetilde{V}}{2}\right)^2 \leq \mathbb{E}\left(\widetilde{X} - \widetilde{V}\right)^2 + \mathbb{E}\left(\widetilde{X} - \widetilde{U}\right)^2$$

[1] Note that this also gives us an immediate proof of $\rho_{UX} \leq 1$. Why?

Or

$$\frac{1}{2}(2-2\rho_{UV}) \leq (2-2\rho_{XV}) + (2-2\rho_{XU})$$

Rearrange to get the desired result.

(d) We can calculate

$$\begin{aligned}
\mathrm{Var}(W) &= \mathrm{Var}\left(2\frac{\sigma_{UX}}{\sigma_X^2}X - U\right) \\
&= 4\frac{\sigma_{UX}^2}{\sigma_X^4}\mathrm{Var}(X) + \mathrm{Var}(U) - 2\left(2\frac{\sigma_{UX}}{\sigma_X^2}\right)\mathrm{Cov}(X,U) \\
&= 4\frac{\sigma_{UX}^2}{\sigma_X^2} + \sigma_U^2 - 4\frac{\sigma_{UX}^2}{\sigma_X^2} \\
&= \sigma_U^2
\end{aligned}$$

Hence $\sigma_{WV} \leq \sigma_W \sigma_V = \sigma_U \sigma_V$. Expanding the LHS,

$$\begin{aligned}
\mathrm{Cov}(W,V) &= \mathrm{Cov}\left(2\frac{\sigma_{UX}}{\sigma_X^2}X - U, V\right) \\
&= 2\frac{\sigma_{UX}}{\sigma_X^2}\sigma_{XV} - \sigma_{UV}
\end{aligned}$$

So we have

$$\begin{aligned}
2\frac{\sigma_{UX}}{\sigma_X^2}\sigma_{XV} - \sigma_{UV} &\leq \sigma_U\sigma_V \\
\implies 2\frac{\sigma_{UX}}{\sigma_X\sigma_U}\frac{\sigma_{VX}}{\sigma_X\sigma_V} - \frac{\sigma_{UV}}{\sigma_U\sigma_V} &\leq 1 \\
\implies 2\rho_{XU}\rho_{XV} - \rho_{UV} &\leq 1
\end{aligned}$$

Rearranging, we get our answer.

□