1. The sets

$$\mathcal{R}_x = \{(x_1, \cdots, x_n) : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$$
$$\mathcal{R}_y = \{(y_1, \cdots, y_n) : 0 \leq y_i \leq 1, \sum_{i=1}^{n} y_i \leq 1\}$$

are related through the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Note that $\mathcal{R}_y$ is the region associated with an n-dimensional unit simplex (look back to Example 3.3.2 from Lab 3). Show that $\text{vol}(\mathcal{R}_x) = \text{vol}(\mathcal{R}_y) = \frac{1}{n!}$.

Proof. Because the transformations are completely linear, the Jacobian matrix $J_{\mathbf{X}\to\mathbf{Y}}$ is exactly the $n \times n$ matrix above. Because the matrix is upper triangular, $\det J_{\mathbf{X}\to\mathbf{Y}} = 1$, the product of the entries along the diagonal. Hence

$$\frac{1}{n!} = \text{vol}(\mathcal{R}_x) = \int_{\mathcal{R}_x} d\mathbf{x} = \int_{\mathcal{R}_y} |\det J_{\mathbf{X}\to\mathbf{Y}}| d\mathbf{y} = \int_{\mathcal{R}_y} d\mathbf{y} = \text{vol}(\mathcal{R}_y)$$

□

2. We say $\mathbf{X}$ follows a uniform distribution *within* the unit n-sphere if

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{V} \cdot \mathbb{I}(x_1^2 + \cdots + x_n^2 \leq 1)$$

for some constant V such that $\int_{\mathbb{R}^n} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = 1$. Use generalized spherical coordinates to show that $V = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$. Geometrically interpret V.

Proof. Notice $f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{V} \cdot \mathbb{I}(r^2 \leq 1) = h(r)$. Hence by generalized spherical coordinates,

$$f_R(r) = h(r) r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} = \frac{1}{V} \cdot \mathbb{I}(r^2 \leq 1) r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}$$

for $r \geq 0$. Since this is a valid density, it must integrate to 1:

$$\begin{aligned} 1 &= \int_0^\infty \frac{1}{V} \cdot \mathbb{I}(r^2 \leq 1) r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} dr \\ &= \int_0^1 \frac{1}{V} r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} dr \\ &= \frac{1}{V} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \left\{ \frac{1}{n} r^n \right\}_0^1 \\ &= \frac{1}{V} \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} \end{aligned}$$

Hence $V = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}$. V is the volume of the n-dimensional unit sphere. □

3. Similarly, we say that $\mathbf{X}$ follows a uniform distribution *on* the unit n-sphere if

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{S} \cdot \mathbb{I}(X_1^2 + \cdots + X_n^2 = 1)$$

for some constant S such that $\int_{\mathbb{R}^n} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = 1$. Show that $S = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$. Geometrically interpret S.

Proof. Similar to problem 3, notice $f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{S} \cdot \mathbb{I}(r^2 = 1) = h(r)$. Hence by generalized spherical coordinates,

$$f_R(r) = h(r) r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} = \frac{1}{S} \cdot \mathbb{I}(r^2 = 1) r^{n-1} \frac{(2\pi)^{n/2}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}$$

for $r \geq 0$. Note that, by definition $\mathbb{P}(R = 1) = 1$, hence the function above is really a pmf, not a pdf. Evaluating at $r = 1$ gives us

$$1 = f_R(1) = \frac{1}{S} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \{r^{n-1}\}_{r=1} = \frac{1}{S} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

Hence $S = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$. S is the surface area of the n-dimensional unit sphere. □

4. Let $\mathbf{X} \sim N(\mathbf{0}, \mathcal{I}_n)$, and define

$$R = \|\mathbf{X}\| = \sqrt{X_1^2 + \cdots X_n^2} \qquad \text{and} \qquad \mathbf{U} = \frac{\mathbf{X}}{R} = \begin{pmatrix} \cos(\Theta_1) \\ \sin(\Theta_1)\cos(\Theta_2) \\ \vdots \\ \sin(\Theta_1)\cdots\sin(\Theta_{n-2})\cos(\Theta_{n-1}) \\ \sin(\Theta_1)\cdots\sin(\Theta_{n-1}) \end{pmatrix}$$

(a) Use generalized spherical coordinates to show the joint distribution for $(R, \mathbf{\Theta})$ factorizes: $f_{R,\mathbf{\Theta}}(r, \boldsymbol{\theta}) = f_R(r) f_{\mathbf{\Theta}}(\boldsymbol{\theta})$. Since $\mathbf{U}$ is only a function of $\mathbf{\Theta}$, conclude that $R \perp\!\!\!\perp \mathbf{U}$.

(b) Show that the density of $\mathbf{U}$ in terms of $\mathbf{\Theta}$ is

$$f_{\mathbf{U}}(\mathbf{u}) = f_{\mathbf{U}}(\mathbf{u}(\boldsymbol{\theta})) = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \frac{1}{\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{p-2})|\sin(\theta_{p-1})|}$$

Proof.

(a) We have

$$\begin{aligned} f_{R,\mathbf{\Theta}}(r, \boldsymbol{\theta}) &= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}r^2} r^{n-1} \sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\cdots\sin(\theta_{n-2}) \\ &= \left\{\frac{r^{n-1}e^{-\frac{1}{2}r^2}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}-1}}\right\}\left\{\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}\sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\cdots\sin(\theta_{n-2})\right\} \\ &= f_R(r) f_{\mathbf{\Theta}}(\boldsymbol{\theta}) \end{aligned}$$

Hence $R \perp\!\!\!\perp \mathbf{\Theta}$, which implies $R \perp\!\!\!\perp \mathbf{U}$.

(b) Denote $\mathbf{U} = (U_1, \cdots, U_n)$. Given the restriction $\|\mathbf{U}\| = 1$, the first $n-1$ components of $\mathbf{U}$ determines U_n up to a sign. When $U_n \geq 0$ (or equivalently, $\theta_{n-1} \in [0, \pi]$), we have $\det J_{\mathbf{U}_{n-1} \to \mathbf{\Theta}}$ to be

$$\begin{aligned} \det \frac{\partial(U_1, \cdots, U_{n-1})}{\partial(\theta_1, \cdots, \theta_{n-1})} &= \det \begin{pmatrix} -\sin(\theta_1) & 0 & \cdots & 0 \\ * & -\sin(\theta_1)\sin(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}) \end{pmatrix} \\ &= (-1)^{n-1}\sin^{n-1}(\theta_1)\sin^{n-2}(\theta_2)\cdots\sin^2(\theta_{n-2})\sin(\theta_{n-1}) \end{aligned}$$

Hence $|\det J_{\mathbf{U}_{n-1} \to \mathbf{\Theta}}| = \sin^{n-1}(\theta_1)\sin^{n-2}(\theta_2)\cdots\sin^2(\theta_{n-2})\sin(\theta_{n-1})$. When $U_n \leq 0$ (or equivalently $\theta_{n-1} \in [\pi, 2\pi]$), we also get the expression above. Hence

$$
\begin{aligned}
f_{\mathbf{U}}(\mathbf{u}) &= f_{\mathbf{\Theta}}(\boldsymbol{\theta})|\det J_{\mathbf{\Theta}\to\mathbf{U}_{n-1}}| \\
&= f_{\mathbf{\Theta}}(\boldsymbol{\theta})|\det J_{\mathbf{U}_{n-1}\to\mathbf{\Theta}}|^{-1} \\
&= \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \frac{1}{\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-2})|\sin(\theta_{n-1})|}
\end{aligned}
$$

□