1. Define the function $\widetilde{d}(X,Y) = \mathbb{E}[|X-Y| \wedge 1]$ (recall $a \wedge b = \min(a,b)$). Prove that $X_n \xrightarrow{\mathcal{P}} X \iff \lim_{n\to\infty} \widetilde{d}(X_n, X) = 0$. **Hint:** Prove that $d(X,Y) \leq \widetilde{d}(X,Y) \leq 2d(X,Y)$, where d is defined in Example 9.2.1.

Solution. Note that $\frac{|X-Y|}{1+|X-Y|} \leq |X-Y|$ and $\frac{|X-Y|}{1+|X-Y|} \leq 1$; hence $\frac{|X-Y|}{1+|X-Y|} \leq |X-Y| \wedge 1$. Take expectation to arrive at $d(X,Y) \leq \widetilde{d}(X,Y)$. For the upper inequality, we examine two cases:

- For $|X-Y| \leq 1$, we have $|X-Y| \wedge 1 = |X-Y|$. Then

$$\begin{aligned} & |X-Y| \leq 1 \\ \iff & 1 \qquad\quad \leq \frac{2}{1+|X-Y|} \\ \iff & |X-Y| \leq \frac{2|X-Y|}{1+|X-Y|} \end{aligned}$$

- For $|X-Y| \geq 1$, we have $|X-Y| \wedge 1 = 1$. Also, $\frac{|X-Y|}{1+|X-Y|} \geq \frac{1}{2}$, hence $2\frac{|X-Y|}{1+|X-Y|} \geq 1$.

So, no matter what, $|X-Y| \wedge 1 \leq 2\frac{|X-Y|}{1+|X-Y|}$. Take expectations to get $\widetilde{d}(X,Y) \leq 2d(X,Y)$. Then we can manipulate the inequalities in Example 9.2.1 to get

$$\frac{\varepsilon}{1+\varepsilon}\mathbb{P}(|X_n - X| > \varepsilon) \leq \widetilde{d}(X,Y) \leq 2[\varepsilon + \mathbb{P}(|X_n - X| > \varepsilon)]$$

□

2. let $X_i \overset{\text{iid}}{\sim} F$ and F satisfies $\lim_{x\to\infty} x^2[1-F(x)] = 0$. Prove that $\max_{1\le i\le n} \frac{X_i}{\sqrt{n}} \xrightarrow{\mathcal{P}} 0$.

Solution. First, we have

$$\begin{aligned}
\mathbb{P}\left(\left|\max_{1\le i\le n} \frac{X_i}{\sqrt{n}}\right| > \varepsilon\right) &= \mathbb{P}\left(\max_{1\le i\le n} X_i > \varepsilon\sqrt{n}\right) + \mathbb{P}\left(\max_{1\le i\le n} X_i < -\varepsilon\sqrt{n}\right) \\
&\le \left\{1 - \prod_{i=1}^{n} \mathbb{P}(X_i \le \varepsilon\sqrt{n})\right\} + \prod_{i=1}^{n} \mathbb{P}(X_i \le -\varepsilon\sqrt{n}) \\
&= 1 - F^n(\varepsilon\sqrt{n}) + F^n(-\varepsilon\sqrt{n})
\end{aligned}$$

$F^n(-\varepsilon\sqrt{n}) \to 0$ by properties of cdf, so it suffices to show that $1 - F^n(\varepsilon\sqrt{n}) \to 0$.

$$\begin{aligned}
1 - F^n(\varepsilon\sqrt{n}) &= \left(\sum_{k=0}^{n-1} F^k(\varepsilon\sqrt{n})\right)[1 - F(\varepsilon\sqrt{n})] \\
&\le n[1 - F(\varepsilon\sqrt{n})] \\
&= \frac{x^2}{\varepsilon^2}[1 - F(x)] && x \mapsto \varepsilon\sqrt{n} \\
&\to 0
\end{aligned}$$

□

3. Let $X_i \overset{\text{iid}}{\sim} \text{Exp}(1)$. Show that $\mathbb{P}(X_n > \alpha \log(n) \text{ i.o.}) = \mathbb{I}(\alpha \leq 1)$. **Hint:** Borel-Cantelli + the integral test for convergence.

Solution. By Borel-Cantelli, it suffices to consider evaluating

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > \alpha \log(n)) = \sum_{n=1}^{\infty} e^{-\alpha \log(n)} = \sum_{n=1}^{\infty} n^{-\alpha}$$

The finiteness (or infiniteness) of this sum is evaluated through a corresponding integral:

$$\int_1^{\infty} x^{-\alpha} dx = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha}\Big|_{x=1}^{\infty} & \alpha \neq 1 \\ \log(x)\Big|_{x=1}^{\infty} & \alpha = 1 \end{cases}$$

which equals ∞ when $\alpha \leq 1$ and is finite when $\alpha > 1$. Hence $\mathbb{P}(X_n > \alpha \log(n) \text{ i.o.}) = 1$ when $\alpha \leq 1$ and $= 0$ when $\alpha > 1$, which is nicely encoded by the indicator function. □

4. Let $X_1, X_2, \cdots$ be positive, iid RV's with common density $f(x)$, satisfying $\lim_{x \downarrow 0} f(x) = \lambda$. Prove that $n \min(X_1, \cdots, X_n) \xrightarrow{\mathcal{D}} \text{Exp}(\lambda)$.

Solution. Define $Y = n \min(X_1, \cdots, X_n)$. We find that

$$S_Y(y) = S^n\left(\frac{y}{n}\right)$$

where $S_Y(y) = \mathbb{P}(Y > y)$ and $S(x) = \mathbb{P}(X_i > x)$. By Taylor expansion,

$$S\left(\frac{y}{n}\right) = S(0) + S'(\xi)\frac{y}{n} = 1 - f(\xi)\frac{y}{n}$$

for some $\xi \in (0, \frac{y}{n})$. Hence

$$S_Y(y) = \left(1 - f(\xi)\frac{y}{n}\right)^n$$

If $z_n \to z$, then we have the standard calculus fact $\lim_{n\to\infty}(1 - \frac{z_n}{n})^n = e^{-z}$. Hence,

$$\lim_{n\to\infty} S_Y(y) = e^{-f(0)y} = e^{-\lambda y}$$

which is the survival function of $\text{Exp}(\lambda)$. $\square$

5. Let $X_i \overset{\text{iid}}{\sim} \text{Unif}(0,1)$ and set $Y_n = \prod_{i=1}^n X_i$. Define the random function $f(z) = \sum_{n=1}^\infty Y_n z^n$, and let R be the radius of convergence of f. Show that $\mathbb{P}(R = e) = 1$, where $e = 2.718\cdots$. **Hint:** Recall radius of convergence is defined as $R = \frac{1}{\limsup |Y_n|^{1/n}}$, and you may use the fact that Y_n has density $f_{Y_n}(y) = \frac{\log^{n-1}(1/y)}{(n-1)!}$.

Solution. It suffices to show that $\mathbb{P}(\lim_{n\to\infty} |Y_n|^{1/n} = e^{-1}) = 1$, or $|Y_n|^{1/n} \overset{\text{a.s.}}{\to} e^{-1}$. Let's first find the density of $|Y_n|^{1/n}$:

$$f_{|Y_n|^{1/n}}(t) = \frac{\log^{n-1}(1/t^n)}{(n-1)!} nt^{n-1} = \frac{n^n}{(n-1)!}(t\log(t^{-1}))^{n-1}$$

Next, it suffices to prove complete convergence, since it implies almost-sure convergence:

$$\sum_{n=1}^\infty \mathbb{P}(||Y_n|^{1/n} - e^{-1}| > \varepsilon) = \sum_{n=1}^\infty \int_{\{t\in(0,1):|t-e^{-1}|>\varepsilon\}} \frac{n^n}{(n-1)!}(t\log(t^{-1}))^{n-1}dt$$

To bound the expression, we will use the fact that $\int_S f(x)dx \le \text{vol}(S)\max_{x\in S} f(x)$. Specifically, let $S = S_\varepsilon = \{t \in (0,1) : |t-e^{-1}| > \varepsilon\} = (0, -\varepsilon+e^{-1}) \cup (\varepsilon+e^{-1}, 1)$, so $\text{vol}(S_\varepsilon) = 1-2\varepsilon$. We have

$$\begin{aligned}\sum_{n=1}^\infty \mathbb{P}(||Y_n|^{1/n} - e^{-1}| > \varepsilon) &\le \sum_{n=1}^\infty \left\{(1-2\varepsilon)\max_{t\in S_\varepsilon} \frac{n^n}{(n-1)!}(t\log(t^{-1}))^{n-1}\right\} \\ &= (1-2\varepsilon)\sum_{n=1}^\infty \left\{\frac{n^n}{(n-1)!}\max_{t\in S_\varepsilon}(t\log(t^{-1}))^{n-1}\right\} \qquad (\star)\end{aligned}$$

To evaluate the sum $(\star)$, we need the following lemma:

Lemma: $\sum_{n=1}^\infty \frac{n^n}{(n-1)!}x^{n-1}$ converges if and only if $|x| < e^{-1}$.
Proof. Recall Stirling's approximation gives us

$$\lim_{n\to\infty} \frac{n^{n+\frac{1}{2}}e^{-n}}{n!} = \frac{1}{\sqrt{2\pi}}$$

Hence,

$$\frac{n^n}{(n-1)!}x^{n-1} = \left[\frac{n^{n+\frac{1}{2}}e^{-n}}{n!}\right]\left[n^{\frac{1}{2}}e^n x^{n-1}\right]$$

The expression in the first bracket is in a bounded neighborhood of $\frac{1}{\sqrt{2\pi}}$, so it suffices to evaluate the convergence of $\sum_{n=1}^\infty n^{\frac{1}{2}}e^n x^{n-1}$, which can be easily checked to converge for $|x| < e^{-1}$.

Next, define $x(t) = t\log(t^{-1})$ and $x = \max_{t \in S_\varepsilon} x(t)$. It is easy to verify that $x(t)$ is increasing on $(0, e^{-1})$ and decreasing on $(e^{-1}, 1)$, with maximum $x(e^{-1}) = e^{-1}$. But clearly $e^{-1} \notin S_\varepsilon$, we have $0 < x(t) < e^{-1}$ for any $t \in S_\varepsilon$, hence $|x| < e^{-1}$. Applying the lemma, $(\star)$ converges. $\square$